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Correlation Redistribution by Causal Horizons

Redistribuição de Correlação devido a Horizontes Causais

Dissertação apresentada ao Instituto de Física Gleb Wataghin da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Mestre em Física, na área de Física

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I dedicate this work to my father

Waldir de Gioia

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Resumo

Em 1975 Stephen Hawking deduziu que devido a efeitos da mecânica quântica um buraco negro formado via colapso gravitacional irá emitir um espectro térmico de partículas [16]. Em 1976 William Unruh inspirado por esse resultado, descobriu que um observador de Rindler no espaço-tempo de Minkowski, vivendo em sua região de Rindler, percebe o vácuo de Minkowski como um banho térmico [36] e, por conta da região de Rindler aproximar a geometria próxima do horizonte de eventos de um buraco negro eterno de Schwarzschild, o vácuo para um observador em queda livre seria percebido como um banho térmico pelo observador de Schwarzschild. Trabalhando com um exemplo simples que foi vastamente usado na literatura para esse tipo de investigação [15, 22, 10] e empregando métodos de teoria de informação quântica, calculamos a correlação clássica, quântica e total entre os subsistemas observados por um observador de Minkowski e um observador de Rindler à esquerda ou à direita; e também por um observador em queda livre e um observador de Schwarzschild à esquerda ou à direita na região próxima ao horizonte. Conseguimos calcular o emaranhamento de formação para o estado experimentado pelos observadores de Rindler à esquerda e à direita, e vemos que este sinaliza a redistribuição de correlação devido a presença do horizonte causal. Concluímos que esse exemplo simples mostra que um horizonte causal de Rindler ou Schwarzschild redistribui correlações por intermédio do emaranhamento entre as duas partes causalmente desconexas do sistema.

Abstract

In 1975 Stephen Hawking derived that, due to quantum mechanical effects, a black hole formed by gravitational collapse will emit a thermal spectrum of particles [16]. In 1976 William Unruh inspired by this discovered that one Rindler observer in Minkowski spacetime, living on its Rindler wedge, perceives the Minkowksi vacuum as a thermal bath [36] and, since the Rindler wedge approximates the near-horizon geometry of an eternal Schwarzschild black hole, the vacuum state for a free-falling observer would be perceived by as a thermal bath for a Schwarzschild observer. Working with a simple example that has been vastly used in the literature for this kind of investigation [15, 22, 10] and employing quantum information methods, we compute classical, quantum and total correlations between the subsystems observed by a Minkowski and either a left or right Rindler observer; and also by a free-falling and either a left or right Schwarzschild observer. We are able to compute the entanglement of formation for the state probed by the left and right Rindler observers and left and right Schwarzschild observers, and following the methods of [19, 14] we see that it signals the correlation redistribution imparted by the presence of the causal horizon. We conclude that this simple example shows that a causal horizon like the Rindler or Schwarzschild horizons redistribute correlations by means of entanglement between the two causally disconnected parts of the system.

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1 Introduction

Gravitational phenomena is currently best described at classical level by Einstein's theory of General Relativity. This theory, first published in 1915, has passed decisive tests in recent years, including LIGO's gravitational wave detection published in 2016 [1] and the first picture of a black hole published in 2019 [2]. While General Relativity is appropriate to describe the large scale structure of spacetime of cosmological and astrophysical interest, the fundamental constituents of matter and their interactions are better described by Quantum Mechanics, and more specifically, by relativistic Quantum Field Theories. However, there are reasons to believe that at a fundamental level, gravity should also admit a quantum description which is yet unknown. In fact Quantum Gravity has been for many years now a very important research topic in theoretical physics. Today there are many proposals for a possible Quantum Gravity theory, the two more well-known theories being Superstring Theory [27, 28, 6], and Loop Quantum Gravity [30, 8]. Those theories have open issues that are actively investigated and up to this point there are no possible tests since Quantum Gravity is believed to be required at the Planck scale, which requires a huge amount of energy, much more than those currently attainable in accelerators like the LHC (Large Hadron Collider).

Since a quantum theory of the gravitational field is still a work in progress in the Physics community, there is a secondary avenue of investigation, Quantum Field Theory in Curved Spacetimes - a semiclassical approach on which all matter and all other interactions apart from gravity are described quantum mechanically in the framework of Quantum Field Theory, whereas gravity is described by classical General Relativity. The Einstein Field Equations are then replaced by the semiclassical Einstein Field Equations on which the source of gravitational field is taken to be the mean value of the energy-momentum tensor. Of course, one can also neglect the backreaction on spacetime and thus simply consider the gravitational field as a classically generated background.

There are several reasons why that would be a reasonable procedure. First of all, this should be a reasonable classical limit just for gravity. Second, this allows one to investigate situations on which quantum mechanics and general relativity are both important but *quantum* gravity *per se* may not be important, i.e., when one is far from the Planck scale but still on strong gravitational fields. One such situation is near a black hole.

It so happens that in 1975 Stephen Hawking published the famous paper *"Particle Creation by Black Holes"* in which he argued that a classical black hole formed by gravitational collapse emits a thermal flux of elementary particles at Hawking temperature, in natural units:

$$T_H = \frac{1}{8\pi M}. \tag{1.1}$$

In fact, Hawking showed that the mean value of the number of particles at any given frequency observed at late times by an asymptotic inertial observer would match the mean value of a thermal ensemble at that temperature. In later years others have shown that the state of these particles is in fact a thermal state and moreover that the outgoing flow of energy is in fact present on the asymptotic regions (c.f. [13] for a complete review).

Inspired by this result, in 1976, William Unruh investigated a similar situation in Minkowski spacetime. He considered uniformly accelerated observers which are eternally accelerating, the so-called Rindler observers. These observers are analogous to observers outside of a Schwarzschild black hole standing still at constant radial coordinate and they experience a similar causal horizon due to their state of motion and the causal structure of spacetime.

What Unruh has found is that if a quantum field is on the vacuum state for the inertial observers, a Rindler observer will not experience a vacuum state, but rather a thermal state at Unruh temperature

$$T_U = \frac{a}{2\pi}, \tag{1.2}$$

where a is the acceleration of the observer.

What may seem remarkable at first, is that the vacuum state, which is itself a pure state, when observed by the Rindler observer, is described by a mixed thermal state. This, however, occurs because the Rindler observer is restricted to live on the Rindler wedge and hence has no access to all existent degrees of freedom. Put differently, restricting attention to a causally bounded region one looses information. It should therefore be interesting to look at this situation from a quantum information theory perspective and try to make this idea more precise and quantitative.

In fact, in [15] one of the first steps towards that direction were taken. The authors addressed the following problem: a relativistic real Klein-Gordon field is in a two-mode state with at most one excitation on each mode. Each mode is observed by one

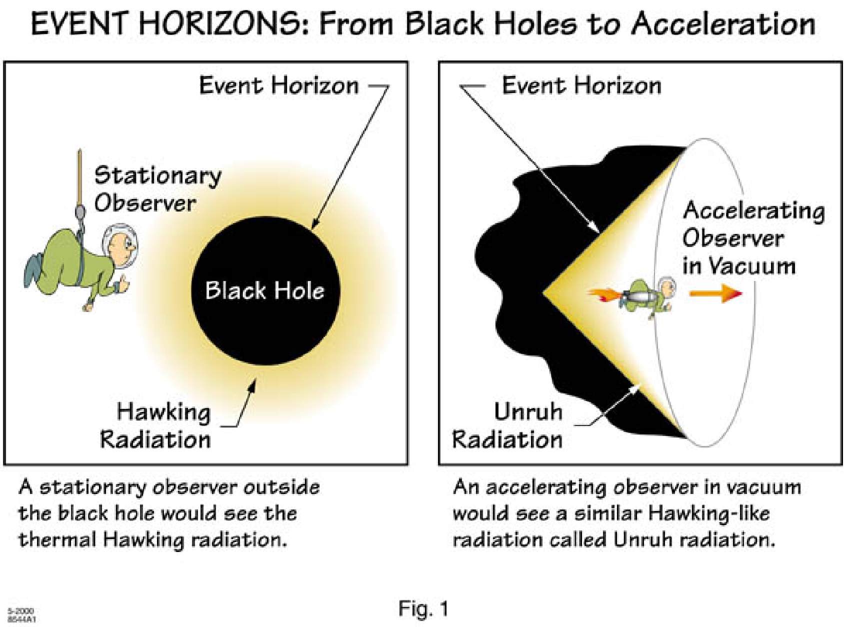

FIGURE 1.1: Illustrative picture of the Unruh and Hawking effects. Copyright: Stanford Linear Accelerator Center https://home.slac.stanford.edu/pressreleases/2000/20000606.htm

inertial observer, Alice and Bob, and from their perspective it is a maximally entangled state. For concreteness the authors took the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_i|0\rangle_i + |1\rangle_j|1\rangle_j), \tag{1.3}$$

where i and j label two radiation modes with frequencies ω_i and ω_j.

Now, one Rindler observer (called Rob by the authors, which is supported on the right Rindler wedge), observes the mode j. Now considering the new bipartition, between Alice and Rob, how the entanglement changes compared to the initial bipartition, between Alice and Bob?

To answer the question the authors used a measure of entanglement, the logarithmic negativity, which is in general easier to compute than other entanglement measures. Secondly, they invoked the so-called "single mode approximation" to say that the Minkowski one-particle state $|1\rangle_j$ expands on the Rindler basis in terms of the same single frequency ω_j. Using these methods, what the authors found was that the causal horizon would impart one entanglement degradation the closer the Rindler observer was to it, i.e., the higher its acceleration.

Years later, the problem was revisited in [23, 22]. First of all the author revised the single-mode approximation. This approximation was in fact not really valid for Minkowski modes, but held exactly for the so-called Unruh modes which were introduced earlier by Unruh in his original derivation of the thermality of the Minkowski vacuum for Rindler observers. In that case, the older results on entanglement near black holes were reinterpreted. Furthermore, the author considered a more complete analysis. A central point of the discussion is that there are two causally disconnected Rindler wedges. This leads people to suppose and formally derive that in the Rindler quantization the Hilbert space is of the tensor product form $\mathscr{H}_I \otimes \mathscr{H}_{II}$, each factor representing degrees of freedom of each wedge,[1]. Using this each Minkowski basis state should expand in a combination of tensor products.

So the pure state, which from the perspective of two inertial observers Alice and Bob was bipartite, from the perspective of an inertial and a Rindler observers Alice and Rob, is in fact tripartite: one complementary third part appears on the left Rindler wedge which could be observed by a complementary observer to Rob, which the authors called AntiRob. The authors computed the negativity (not to be confused with the logarithmic negativity used earlier) and the mutual information for all possible bipartitions of the Minkowski-Rindler system, namely Alice-Rob, Alice-AntiRob and Rob-AntiRob, ariving at a more complete analysis of the phenomenon. Furthermore the authors took the step of taking the analysis to the Schwarzschild spacetime case.

Later in [10] using the same setup it was computed the quantum discord of the bipartition Alice-Rob: one measure of the total quantum correlations which includes

[1]This is not, considering mathematical rigor, true [42, 17], but leads to the correct results nonetheless which may be rigorously justified.

more than just entanglement. He then remarked that the previous works had interpreted that, since the negativity decreased to zero as the observer was closer to the causal horizon, in that regime only classical correlations would remain in the state. Noticing that the quantum discord didn't decrease to zero, but to a finite value as one approaches the causal horizon, the author considered that not all quantum correlations would cease to exist in this regime as previously thought.

Throughout this dissertation we shall review in detail all of these topics. Starting from the necessary standard concepts of General Relativity, Quantum Mechanics and Quantum Information, we review the ideas of Quantum Field Theory in Curved Spacetimes and in particular the details of the derivation of the Unruh effect. We then study the problem of correlation redistribution due to the presence of causal horizons using the same setup we briefly outlined above. We employ the method of [10] to compute not just the quantum discord of the Alice-Rob bipartition, but both classical correlation and quantum discord for both the Alice-Rob and Alice-AntiRob bipartitions, which are the ones for which the method applies.

We recall that classical correlations and quantum discord can be interpreted as locally acessible information and locally inaccessible information respectively, this being nothing more than a change in perspective regarding interpretation that makes it clearer the occurrence of a redistribution of correlations between the two bipartitions.

Finally we employ the methods of [19] to compute the entanglement of formation of the Rob-AntiRob bipartition out of the locally accessible information obtained in the previous step. Unlike negativity or logarithmic negativity, the entanglement of formation is a more appropriate measure of entanglement in the sense that, for instance, it reduces to the entanglement entropy for pure states, furthermore, it allows us a direct interpretation in terms of informational quantities.

In [22] the author mentions that there is not much interest in the entanglement for this bipartition because there is no classical communication allowed between the two observers, rendering the entanglement useless as a resource. In fact, this is true, this entanglement cannot be employed as a resource for any quantum communication task. Nonetheless, entanglement of formation holds special relation to locally acessible and locally inacessible information as has been advocated in [19, 14] which allows to interpret it as signaling a correlation redistribution in the system.

So the entanglement of formation for the state probed by the two observers on each Rindler wedge signals the redistribution on the information stored in correlations and that redistribution can be explicitly seem by plotting the locally acessible and locally inacessible informations for both bipartitions. Compared to the Alice-Bob state, the greater the acceleration the more the correlations of the Alice-Rob subsystem decrease and the more the correlations of the Alice-AntiRob increase. This quantifies the redistribution of information imparted by the causal horizon. We work out the details in the Minkowski setup, and using the methods of [22] we carry these over to the Schwarzschild setup. This dissertation is organized as follows:

- Chapter 2: We review the elements of the theory of general relativity which are related to the Unruh effect;
- Chapter 3: We review the elements of quantum mechanics that allows us to clearly talk about correlations and, in particular, quantum entanglement;
- Chapter 4: We review quantum field theory in curved spacetimes and the Unruh effect.
- Chapter 5: We study the correlation redistribution imparted by a causal horizon in a maximally entangled two-mode state of a neutral scalar field:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_i|0\rangle_j + |1\rangle_i|1\rangle_j) \tag{1.4}$$

when the part that observes the mode of frequency ω_j in fact experiences a causal horizon.

2

General Relativity

In this chapter our aim is to review the basics of General Relativity required for the forthcoming chapters and in the process to establish the notation and conventions in place. The main objective is to explicit the structure of the maximal extension of the Schwarzschild solution representing an eternal black hole which consists of two causally disconnected regions isolated from each other by causal horizons. This is qualitatively the same structure probed by a Rindler observer, even though no real black hole exists in that case. Moreover, the Rindler geometry approximates the Schwarzschild one in the near-horizon region. In that way, this is the basic background structure that we wish to consider to discuss the correlation redistribution.

2.1 Spacetime, observers and reference frames

General Relativity is a theory based on Lorentzian geometry. The basic terminology and notation of said geometric methods are summarized in Appendix A. With these geometric methods we can define what *spacetime* is:

Definition 2.1.1. A **spacetime** is a pentuple $(M, g, \nabla, \epsilon, \uparrow)$ where (M, g) is a smooth orientable Lorentzian manifold, ∇ is its Levi-Civita connexion, ϵ is a volume form for M and $\uparrow$ is a time orientation for (M, g), defined by a smooth nowhere vanishing timelike vector field $X_T \in \sec TM$.

The structure $(\nabla, \epsilon, \uparrow)$ will be left implicit and we shall always talk about a spacetime just as (M, g). In our conventions for the signature of the metric, the time-orientation is defined so that whenever $v \in T_{\mathfrak{e}}M$ is given, it is said to be future-directed if $g(X_T, v) <$ 0 and past-directed if $g(X_T, v) > 0$. A curve is likewise said to be future-directed when all its tangent vectors are future-directed and said to be past-directed when all its tangent vectors are past-directed.

Next we recall some concepts related to observers and reference frames as correctly made rigorous by Sachs and Wu in their General Relativity book [31]. We shall now follow closely this reference. We first define an observer:

Definition 2.1.2. Let (M, g) be a spacetime. An observer is a timelike, future-directed worldline $\gamma : I \subset \mathbb{R} \to M$ such that $g(\gamma', \gamma') = 1$.

Definition 2.1.3. Let (M, g) be a spacetime and $z \in M$ an event. An instantaneous observer at z is a tangent vector $(z, Z) \in T_z M$ on the tangent space that is timelike, future-directed and satisfies $g(Z, Z) = 1$.

Definition 2.1.4. Let (M, g) be a spacetime, (z, Z) an instantaneous observer. Let $\operatorname{span} Z$ be the one-dimensional timelike subspace of the tangent space $T_z M$ spanned by Z and $Z^\perp$ its orthogonal complement with respect to the metric g_z at said event. We call $T = \operatorname{span} Z$ the observer's local time axis and $R = Z^\perp$ its local rest space.

Definition 2.1.5. Let (M, g) be a spacetime, (z, Z) an instantaneous observer with orthogonal decomposition $T_z M = T \oplus R$. We define, respectively, the temporal and spatial projections $\Pi_T : T_zM \to T$ and $\Pi_S : T_zM \to R$ so that any $X \in T_zM$ is $X = \Pi_T(X) + \Pi_S(X)$ uniquely.

The notion captured by the above definitions is that the local rest space is the "available spatial directions in three-dimensional space" perceived by the observer in question. In particular take note that the projections can be determined easily. Since $\Pi_S(X) \in R$ we have that $g(Z, \Pi_S(X)) = 0$. This implies that

$$g(Z, X) = g(Z, \Pi_T(X) + \Pi_S(x)) = g(Z, \Pi_T(X)). \tag{2.1}$$

On the other hand, T is one dimensional, spanned by Z, hence $\Pi_T(X) = eZ$. This implies that

$$g(Z, X) = g(Z, eZ) = e, \tag{2.2}$$

since g is bilinear and $g(Z, Z) = 1$. Thus we have

$$\Pi_T(X) = g(Z, X)Z, \quad \Pi_S(X) = Z - g(Z, X)Z. \tag{2.3}$$

We next capture the idea of relative velocity between two observers.

Definition 2.1.6. Let (M, g) be a spacetime, (z, Z) an instantaneous observer and (z, W) another instantaneous observer, meeting at the same event. Decompose $W = eZ + p$ as discussed above. We define the **Newtonian velocity** of W with respect to Z as p/e.

Definition 2.1.7. Let (M, g) be a spacetime, (z, Z) an instantaneous observer and (z, W) another instantaneous observer, meeting at the same event. We say that (z, Z) and (z, W) are **comoving** when the Newtonian velocity of W with respect to Z is zero. Furthermore, let $\gamma : I \subset \mathbb{R} \to M$ and $\tilde{\gamma} : \tilde{I} \subset \mathbb{R} \to M$ be two observers. Suppose $\gamma(u) = \tilde{\gamma}(\tilde{u}) = z$ so that the instantaneous observers $(\gamma(u), \gamma'(u))$ and $(\tilde{\gamma}(\tilde{u}), \tilde{\gamma}'(\tilde{u}))$

meet at the event z. We say that the observers are comoving at z if the corresponding instantaneous observers are comoving at z.

Since we imposed that all observers and instantaneous observers are unit length in the spacetime metric then two instantaneous observers which are comoving are actually represented by the **same** four-vector. Obviously, given two observers, they can be comoving at one event and not at others. When they are comoving at all events, they are equal by the uniqueness theorem for solutions of ordinary differential equations. Notice here that the idea of comoving is the idea of "seeing the other observer at rest with respect to you", as captured by demanding zero Newtonian velocity.

Next we take care of the idea of reference frames. Despite the confusion majorly inspired by the basic treatments of Special Relativity, reference frames **are different than coordinate systems**, albeit related. A reference frame should be thought of as a collection of observers, "sharing a common motion" in some sense to be captured by a precise definition. We define it as follows

Definition 2.1.8. Let (M, g) be a spacetime. A reference frame is a timelike, future-directed vector field $Z : M \to TM$ such that $g(Z, Z) = 1$. The integral lines of Z are observers which we call **observers in** Z.

The idea behind reference frames is that, since observers are extremely local, being capable of observing just on the events they participate, in general cooperation between observers is needed. One auxilary notion can be immediately constructed

Definition 2.1.9. Let (M, g) be a spacetime, Z a reference frame and (z, W) an instantaneous observer on the domain of Z. We say that (z, W) is **comoving** with Z if (z, W) is comoving with the reference frame's instantaneous observer at the event (z, Z).

The notion of time, however, might be different for the observers on a same reference frame. This leads to the idea of synchronizability of reference frames. The underlying idea is: given a reference frame Z, when can the observers in Z use Einstein's synchronization procedure to establish one notion of *time for* Z? This in turn depends on the following

Definition 2.1.10. Let (M, g) be a spacetime and Z be a reference frame. Let $\zeta = g(Z, \cdot)$ be the physically-equivalent one-form. We say that Z is:

1. **Synchronizable** if there are functions $h, t \in C^\infty(M)$, with $h > 0$ and such that $\zeta = -hdt$. The function t is called a time function for Z;

2. **Proper-time synchronizable** if there is a function $t \in C^\infty(M)$ such that $\zeta = -dt$. The function t is called a proper-time function for Z;

3. **Locally synchronizable** if $\zeta \wedge d\zeta = 0$;

4. **Locally proper-time synchronizable** if $d\zeta = 0$;

Synchronizability is really related to the idea of defining "surfaces of simultaneity" for the reference frame. Indeed the surfaces of time function constant are the surfaces of simultaniety, and physically this would mean that the observers in the reference frame are able to agree that all events on such surface occur simultaneously for them. Physically this can be all explained via Einstein's synchronization procedure (c.f. section 5.3 of [29], specially 5.3.1).

Local synchronizability is the same concept, but only on neighborhoods of events. Indeed, Frobenius' integrability theorem guarantees that if $\zeta \wedge d\zeta = 0$ then, for every point, there is a neighborhood on which the distribution of rest spaces is integrable, in the sense that the "surface of simultaniety" does exist on this neighborhood, allowing for the synchronization of clocks. This in turn is the same as saying that on such neighborhood $\zeta = -hdt$. So, as expected, local synchronizability is the same as synchronizability on some neighborhood of each event.

Albeit perhaps clear by now, we stress that the meaning of all of this can be recast as follows: as we pointed out, observers are so local that reference frames should in some sense introduce the idea of cooperation of observers in making observations of events. As we also explained, every observer has its rest space on each event it participates, representing his own local view of the three-dimensional world. Since a reference frame is a collection of observers, a reference frame gives rise to a collection of rest spaces. Roughly speaking, in Differential Geometry, such collection of subspaces of the tangent spaces of a manifold, is called a distribution, and the distribution is said integrable when, intuitively, it is possible to "bring together" these subspaces to form a smooth surface. That is exactly what is happening here. When the reference frame is synchronizable (resp. locally synchronizable), the rest spaces of its observers can be brought together (resp. locally brought together), giving rise to simultaniety surfaces which should represent the instantaneous three-dimensional spaces for the reference frame in question.

Finally we consider coordinate systems. As (M, g) is a smooth manifold, it is part of its definition that it has charts, also known as coordinate systems. A coordinate system by itself is just a way to assign numbers to events in a smooth way, and to do actual computations locally with the well known multivariate analysis, after all, they locally identify the manifold with $\mathbb{R}^n$. In the case of the physical four-dimensional spacetime, of course $n = 4$.

It should be clear now that there is no idea of motion inherent to a coordinate system. On the other hand, coordinate systems are related to reference frames and observers in a specific way. For that matter we define the following concept:

Definition 2.1.11. Let (M, g) be a spacetime and Z a reference frame. We say that a coordinate system (U, x) on the open set $U \subset M$ is a naturally adapted coordinate system to Z if

1. The coordinate system has one timelike and $n - 1$ spacelike coordinates;

2. On the natural basis of vector fields ∂_μ the spacelike components of Z vanish;

Thus if (U, x) is naturally adapted to Z we have that $Z = \alpha \partial_0$. Now since $g(Z, Z) = 1$ we must have $\alpha = g_{00}^{-1/2}$. This implies that in a naturaly adapted coordinate system Z takes the form $Z = g_{00}^{-1/2} \partial_0$.

Take note that this **does not imply that** Z **is locally synchronizable**. The physically equivalent one-form is, on this coordinate system, $\zeta = Z_\mu dx^\mu$ where $Z_\mu = g_{\mu\nu} Z^\nu$. Hence we have

$$Z_\mu = g_{\mu 0} Z^0 = g_{00}^{-1/2} g_{\mu 0}. \tag{2.4}$$

It is then by no means necessary that ζ satisfies the integrability condition $\zeta \wedge d\zeta = 0$. In truth, we notice that **given a reference frame there is always a naturaly adapted coordinate system** [33], so that obviously if the existence of such system implied synchronizability, all reference frames would be synchronizable, which is certainly false. This shows the relation between reference frames and coordinate systems; in one direction, given a reference frame, we can have many coordinate systems naturaly adapted to it, and all of those have one timelike coordinate and all the others spacelike. In the other direction, a coordinate system with a timelike coordinate and all the others spacelike give rise to a reference frame. Indeed one takes then $Z = \pm g_{00}^{-1/2} \partial_0$, the prefactor necessary in order for Z be unit-lenght which is part of our definition of reference frame, and with the plus sign chosen if ∂_0 is future-directed and the minus sign chosen if ∂_0 is past-directed, so that the resulting Z is future-directed.

One easy fact to prove which will allow us to conclude that many reference frames are locally synchronizable is as follows

Proposition 2.1.1. Let (M, g) be a spacetime, Z a reference frame, and (U, x) a naturaly adapted coordinate system to Z such that there are no "time-space" cross terms in the metric tensor, in other words, $g_{\mu 0} = g_{00} \delta_{\mu 0}$. Then Z is synchronizable.

Proof. As we have shown above, $\zeta = Z_\mu dx^\mu$ with $Z_\mu = g_{00}^{-1/2} g_{\mu 0}$. In the case of the hypothesis, Z_μ is zero unless $\mu = 0$. This in turn implies that $\zeta = g_{00}^{1/2} dx^0$. By the definition of synchronizability, it follows that Z is synchronizable. □

It turns out that many reference frames can be defined as we have pointed out, by starting with a coordinate system with a timelike coordinate and all the others spacelike, and then taking the normalized timelike coordinate basis field as the reference frame. More than that, it is usually the case that when this is done so, the above hypothesis are obeyed. In these cases, we can immediately conclude the synchronizability of the reference frame.

Finally, we mention an important definition, that of a stationary reference frame:

Definition 2.1.12. Let (M, g) be a spacetime. A reference frame Z on $U \subset M$ is said to be **stationary** if there is a positive function $f : M \to \mathbb{R}$ such that fZ is a Killing vector field. The reference frame is further said to be **static** if it is stationary and locally synchronizable.

The definition is clear. An observer on the stationary reference frame Z doesn't perceive change in the gravitational field during his time evolution. The condition on staticity can be better motivated by a result which says that a reference frame is synchronizable if and only if in a sense it is irrotational [29, 31]. In that case, a static reference frame is further non-rotating. Following [31] we can use these definitions to capture properties of spacetimes;

Definition 2.1.13. Let (M, g) be a spacetime. We say that (M, g) is **stationary** if there is a global stationary reference frame and we say that (M, g) is **static** if there is a global static reference frame.

We shall talk about matter in the next section, but we antecipate that it is clear by the previous discussion that a stationary spacetime can be seen by one generated by one distribution of matter which doesn't change in time. A static reference frame is furthermore generated by a distribution of matter which is non-rotating.

2.2 Matter and Einstein's Field Equations

We now turn to the description of matter in general, and the equation which tells how the matter content of a region of spacetime determines its Lorentzian geometry. In the general situation matter is described by the *energy-momentum tensor*. The key idea of General Relativity is that the energy-momentum tensor sources the gravitational field. This will soon allow for a general description of it.

The equations governing the dynamics of the gravitational field follow from the Einstein-Hilbert action

$$S_{\text{EH}}[g] = \int_M R\epsilon \tag{2.5}$$

where integration is taken over the spacetime manifold M, ϵ is the volume form and R is the Ricci scalar. The Euler-Lagrange equations, obtained by demanding $\delta S = 0$ with respect to variations on the metric g, are then the vacuum Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \tag{2.6}$$

Next suppose that there is matter described by some matter action $S_M[g, \Phi]$ where Φ denotes collectively any form of matter present. The matter action depends on the background spacetime, hence the dependence on the metric. The full gravitation plus matter action will be $S = S_{\text{EH}} + S_M$. The equations of motion of the gravitational field still follow from $\delta S = 0$ under variations with respect to the metric. This, on the other hand, implies that we have the equations [7, 39]:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu}, \tag{2.7}$$

with T one symmetric tensor field defined by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.8}$$

We interpret this as the energy-momentum tensor. In other words: by Einstein's idea of matter sourcing the gravitational field, we can turn things around and *define* the object describing the quantity and flow of matter as the source of the gravitational field when it is coupled to matter.

It is important to understand that the solutions are Lorentzian spacetimes (M, g) such that the equation holds everywhere on M. In particular this means that the topology of M is part of the solution. Put differently, spacetimes which are distinct topologically and still have the same functional form of the metric in some coordinate system still count as distinct solutions since the Lorentzian manifold (M, g) will be different by the very definition of a manifold.

2.3 The Schwarzschild Solution

2.3.1 Spherical Symmetry

We now turn to the discussion of the specific class of spacetimes which are spherically symmetric. A spherically symmetric spacetime is one generated by a spherically symmetric distribution of matter. So for example, either a massive spherical shell, or a massive ball, give rise to a spherically symmetric solution to the Einstein's field equations. The definition of spherically symmetric spacetime is [39]:

Definition 2.3.1. A spacetime (M, g) is said to be spherically symmetric if the rotation group $SO(3)$ acts on it by isometries $\delta : SO(3) \times M \to M$, whose orbits $O_p = \{\delta(g, p) \in M : g \in SO(3)\}$ are homeomorphic to S^2.

Now suppose that the spacetime is further static and that the static timelike Killing field is unique (see Section 2.1 for the definition of static spacetimes). Then it is shown in [39] that there is a coordinate system (t, r, θ, ϕ) in M such that the metric becomes

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.9}$$

To understand this, first recall that orbits of the rotations are spheres. The coordinates (θ, ϕ) are S^2 coordinates carried over to these orbits. The coordinate r is a coordinate with the property that on each such sphere it attains the value $r = \sqrt{A/4\pi}$ where A is the surface area of the sphere. Finally t is a coordinate measuring the time of the static reference frame.

Up to this point we have not solved the Einstein's equations at all. We have presented only the constraint that picks out one family of spacetimes to be studied. The actual solutions are found when this form of the metric is used in the Einstein's field equations with an appropriate source [39]. In particular, in a spherically symmetric spacetime, if there is one region satisfying the vacuum equations, it can be shown that its metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.10}$$

where M can be interpreted as the mass of a central object enclosed by the vacuum region. This is known as the Schwarzschild solution [39, 7]. Notice first that there appears to be two problems with the metric given by Eq. (2.10). The first one is at $r = 2M$. The metric tensor seems to be singular at this surface. It turns out that this is not really the case. This is just due to a bad choice of coordinates and in fact this location can be approached in these coordinates but it is not part of the domain of the chart. Indeed, it is possible to construct a new coordinate system, such as Lemaitre coordinates, on which it is manifest that there is no real singularity at $r = 2M$ [21]. In fact, we shall show this explicitly constructing other sets of coordinates, which achieve the same purpose but are more suitable for our purposes. In this sense, $r = 2M$ is not actually an issue, although it is special as we shall soon discuss. A second problem arises at $r = 0$. This one, *at first*, seems to be also just a poor choice of coordinates. By definition of the radial coordinate, $r = 0$ would correspond to a sphere with zero area. In that setting, the angular part of the chart would loose injectivity there. This would be analogous to the reason why spherical coordinates in $\mathbb{R}^3$ *do not cover the origin*. However, the situation here is more serious. It turns out that it can be shown that there are curvature invariants which actually diverge as $r \to 0$ [39, 7]. For example the Kretschmann scalar defined by

$$K = R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}, \tag{2.11}$$

is one curvature invariant which in the Schwarzschild spacetime is written as

$$K = \frac{48M^2}{r^6}. \tag{2.12}$$

This clearly diverges as $r \to 0$. Such a quantity, being defined as a contraction of two tensors, is intrinsic to the spacetime manifold and its behavior is independent of any coordinate chart used. The fact that it diverges as $r \to 0$ implies that *the spacetime manifold is inextendible to another spacetime containing the desired point*. This means that the point in question is a true singularity of the solution.

2.3.2 Future and Past Horizons

We would like to better understand what happens at $r = 2M$ and to be able to discern if the Schwarzschild coordinates are just inappropriate coordinates there, so that it is possible to describe this locus of points in a more appropriate coordinate system or if this is a true singularity like the one at $r = 0$.

In a sense, the canonical way to answer the above inquiry is to approach the locus $r = 2M$ along physically reasonable worldlines and to ask if along such path one gets to $r = 2M$ at finite affine parameter or not [39]. If the answer is positive, one constructs a coordinate system adapted to these worldlines and then is able to extend the coordinates pass through $r = 2M$.

In our case, a suitable way to do this is by considering ingoing and outgoing radial null geodesics which are paths of ingoing and outgoing massless particles. The locus $r = 2M$ can be approached along geodesics of the two classes in finite affine parameter. Moreover, the actual points in the spacetime manifold which are approached along the ingoing radial null geodesics are different than the ones approached along the outgoing radial null geodesics, so that $r = 2M$ consists of two parts. We shall now make this more precise, referring the reader to C for the details of the construction of these coordinates out of the study of geodesic motion.

As a first step one defines first the so-called *tortoise coordinate* [39, 7, 35] on the region $r > 2M$

$$r_* = r + 2M \ln \frac{r - 2M}{2M}, \quad r > 2M. \tag{2.13}$$

Next, one defines the ingoing Eddington-Finkelstein null coordinate

$$v = t + r_*, \quad r > 2M, \tag{2.14}$$

and considers the coordinate system (v, r, θ, ϕ), called the *ingoing Eddington-Finkelstein coordinate system*. The curves with v, θ, ϕ constant are the aforementioned ingoing radial null geodesics and the radial coordinate r evaluated along these lines gives an affine parameter. In other words: the Schwarzschild radius is an affine parameter along these curves. The metric in this coordinate system becomes

$$g = -f(r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad f(r) = 1 - \frac{2M}{r}, \tag{2.15}$$

and it is clear that its coordinate expression has no singularity at $r = 2M$. Even though the coordinates have been defined on the $r > 2M$, since $r = 2M$ is approached at finite affine parameter, one may include the corresponding points. In other words, the part of the locus $r = 2M$ approached along ingoing radial null geodesics is well represented in terms of these coordinates and we shall denote it $\mathcal{H}^+$ and call it *future event horizon*.

Likewise, one defines the outgoing Eddington-Finkelstein null coordinate

$$u = t - r_*, \quad r > 2M, \tag{2.16}$$

and consider the coordinate system (u, r, θ, ϕ), called the *outgoing Eddington-Finkelstein coordinate system*. Now the u, θ, ϕ constant curves are the outgoing radial null geodesics and again r an affine parameter along them. The metric in this coordinate system is

$$g = -f(r)du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad f(r) = 1 - \frac{2M}{r}, \tag{2.17}$$

and there is also no coordinate singularity at $r = 2M$. The coordinates, originally defined at $r > 2M$ can be extended to include the part of the locus $r = 2M$ approached along the outgoing radial null geodesics. That locus of points in the spacetime manifold is denoted $\mathcal{H}^-$ and called the *past event horizon*.

2.3.3 The Kruskal-Szekeres Extension

A drawback of both ingoing and outgoing Eddington-Finkelstein coordinate systems is that they are naturally adapted to either ingoing or outgoing null radial geodesics, *but not both*. This is not a problem if we want to understand just the local structure of the spacetime under consideration, but if we want to understand it globally then it is indeed a problem. The issue could be remedied, however, by changing to the mixed coordinate system (u, v, θ, ϕ). There is one drawback, however, in the new coordinate system, that even though the radial null geodesics corresponds to coordinate lines of either u or v, the horizons $\mathcal{H}^{\pm}$ are "far away" in the sense that they are approached only asymptotically as $u \to \infty$ or $v \to -\infty$.

This can easily be seen, as follows. Take one ingoing null radial geodesic affinely parameterized by r. It has (v, θ, ϕ) constant along it, and $u = v - 2r_*$. Now take $r \to 2M$, then $r_* \to -\infty$ and $u \to +\infty$. Analogously, for one outgoing null radial geodesic affinely parameterized by r, one has (u, θ, ϕ) constant along it with $v = u + 2r_*$. Then as $r \to 2M$ one has $r_* \to -\infty$ and $v \to -\infty$. In a different way, the spacetime location corresponding to $r = 2M$ is infinitely far away in this chart. Still, as pointed out earlier, we have a clear indication that this is indeed a physical surface in spacetime, because it is approached in *finite affine parameter* along geodesics.

Thus the solution is to construct yet another coordinate system, which pulls $\mathcal{H}^{\pm}$ to finite coordinate values. This can obviously be done in many ways, but there is one that is natural because of a second geometric interpretation it yields. The idea is to define functions U, V which have the property that evaluated along the generators of the past and future horizons will give the corresponding affine parameters. The key computations leading to the definition of such functions are found in section C.2.

With the approach motivated, define the functions $V = e^{v/4M}$ and $U = -e^{-u/4M}$. They still characterize ingoing and outgoing radial null geodesics by either V constant or U constant. Apart from that, when evaluated on top of the generators of the surfaces $\mathcal{H}^{\pm}$ they coincide with the affine parameters. So it is obviously well defined in that surface, having actual geometric meaning there.

A minor comment is necessary to explain where the minus sign comes from. If one simply inverted the relation between u and the affine parameter, one would expect to obtain $\tilde{U} = e^{-u/4M}$, so why does one include that minus sign? The most lucid reason to do so is to recall the following. On ingoing null geodesics, v is constant while r is the affine parameter. Because of that $u = v - 2r_*$ can be used as parameter and so does $\tilde{U}$ since it is a function of u alone. In that case along such curve $\tilde{U} = e^{-v/4M}e^{r_*/2M}$. Now notice that as r increases along the curve, $\tilde{U}$ increases as well. But along ingoing null radial geodesics, the increase of r means one is heading to the past. In that sense, the increase of $\tilde{U}$ correspond to progress towards the past. Reversing the sign, by taking $U = -\tilde{U}$ now means the opposite: increase in U means progress into the future and so we include the minus sign because of this convenience.

This procedure defines a chart (U, V, θ, ϕ) which is called the *Kruskal-Szekeres double-null coordinate chart*. Interestingly, we observe one property of said coordinate system. The coordinate U is restricted to $(-\infty, 0)$ and the coordinate V is restricted to $(0, +\infty)$. Together they cover the exterior region $r > 2M$ of the spacetime with the horizon at $r = 2M$ corresponding to either $U = 0$ or $V = 0$, which is equivalent to the condition $UV = 0$. Actually, we have on the exterior region,

$$UV = -e^{-u/4M}e^{v/4M} = -e^{(v-u)/4M}, \tag{2.18}$$

but $(v-u)/2 = r_*$ and hence

$$UV = -e^{r_*/2M} = -e^{r/2M}\frac{|r-2M|}{2M} = \left(1 - \frac{r}{2M}\right)e^{-r/2M}. \tag{2.19}$$

From Eq. (2.19) we see explicitly that the horizon corresponds to $UV = 0$. Yet, the fact that U and V do not cover the whole real axis means that this spacetime contains geodesics which are not complete. The reason is that their coordinate lines *are geodesics*, and they reach the end of the chart at finite affine parameter. Indeed both attain $U, V = 0$ when $r \to 2M$ which is a finite vaue of the affine parameter. In that sense, this spacetime is *geodesically incomplete*. We can extend this spacetime by letting the coordinates U, V run over all $\mathbb{R}$, without reaching the physical singularity at $r = 0$, and analitically continuing the metric tensor to these values.

The spacetime obtained is called the Kruskal-Szekres maximal extension of the Schwarzschild spacetime. It is still geodesically incomplete because of the $r = 0$ singularity which we argued to be a true geometrical singularity. On the other hand, it is *inextendible*, so that we have extended it as far as possible compatibly with the singularity at the origin [20].

In this spacetime we have $\mathcal{H}$ fully defined by the condition $UV = 0$. Furthermore, the Kruskal-Szekeres double null chart covers all of it. On the other hand, the extension procedure adds in one *paralell exterior region*, which looks like a copy of the $r > 2M$ exterior region causally disconeccted from the initial one. In particular, one defines Schwarzschild coordinates there by means of the Kruskal-Szekeres double null coordinates. Similarly it adds more pieces to $\mathcal{H}$ which are in turn boundaries of this new exterior region.

To draw one diagram of the obtained spacetime we create new coordinates T and X by

$$T = \frac{U + V}{2}, \quad X = \frac{V - U}{2}. \tag{2.20}$$

Next we notice that the physical singularity at $r = 0$ corresponds naturally to $UV = 1$ and hence to $T^2 - X^2 = 1$. So the full maximally extended Schwarzschild solutions is described by the full range of coordinates U, V with $UV < 1$ or else by the full range of coordinates T, X with $T^2 - X^2 < 1$. This gives the diagram shown in Fig. (2.1):

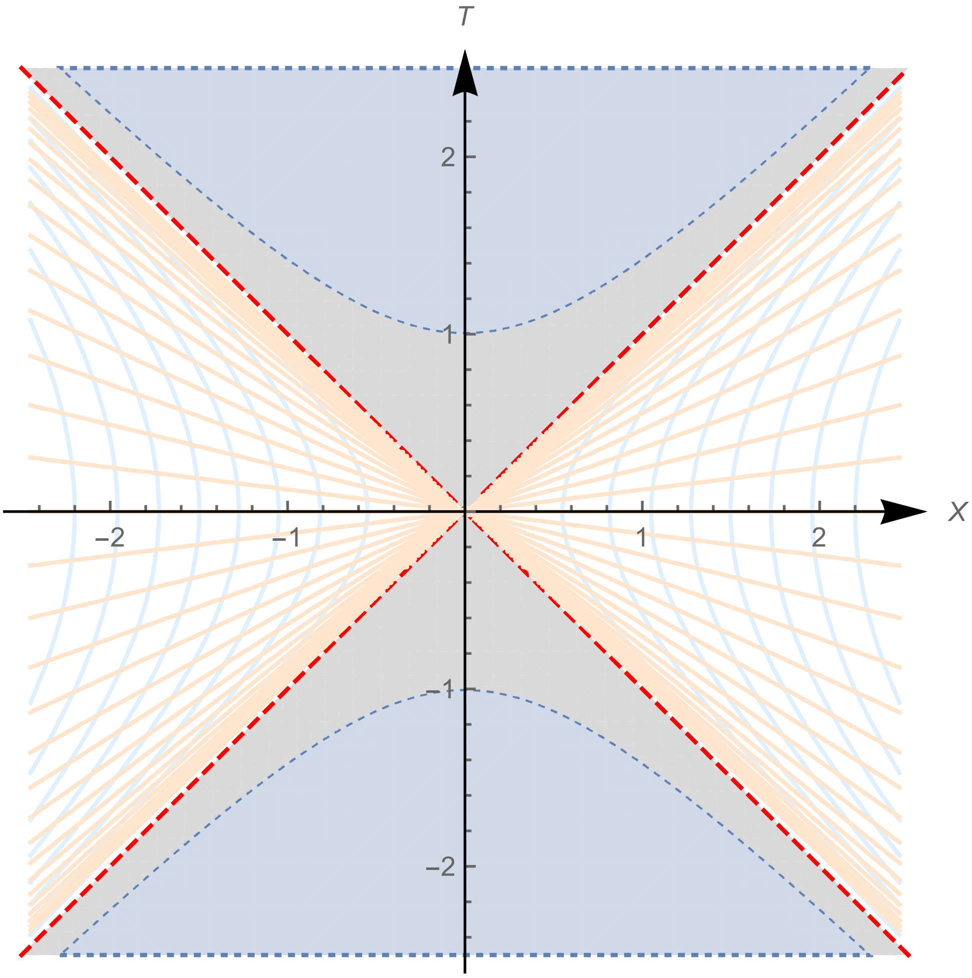

FIGURE 2.1: The Kruskal-Szekeres diagram. The physical singularity is the blue dotted line, while the blue region is not part of spacetime. The horizons are the red dashed lines and the grey regions are the black hole and white hole interior. The light orange lines are lines of constant t and the light blue lines are lines of constant r.

2.4 Near-Horizon geometry and Rindler Spacetime

Now we finally shall consider one near-horizon approximation on which we shall obtain the Rindler geometry. To do so, fix a Schwarzschild observer at (r_0, θ_0, ϕ_0). It is described in coordinates by the curve $(t, r_0, \theta_0, \phi_0)$ and has tangent vector ∂_t which is not normalized since it is not propertime parameterized. Still, the propertime may be derived with ease:

$$\tau(t) = \int^t \sqrt{-g_{tt}}dt' = \int^t f(r_0)^{1/2}dt' = f(r_0)^{1/2}t, \tag{2.21}$$

where we could take $f(r_0)$ out of the integral, since it depends just on the radius which is fixed along the worldline of the observer.

Define the coordinate $\tau_0 = f(r_0)^{1/2}t$. It is just a naive re-scaling of the time coordinate. It satisfies $dt = f(r_0)^{-1/2}d\tau_0$ from which we can compute the metric

$$ds^2 = -\frac{f(r)}{f(r_0)}d\tau_0^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.22}$$

Next we change the spatial coordinate. One defines

$$\frac{z^2}{8M} = r - 2M. \tag{2.23}$$

We can now derive $f(r)$ in terms of the new coordinate z. It is

$$f(r) = 1 - \frac{2M}{r} = \frac{r-2M}{r} = \frac{z^2/8M}{z^2/8M + 2M}. \tag{2.24}$$

To simplify the equation, we divide numerator and denominator by $2M$ so that in terms of $\kappa = 1/4M$ it is

$$f(r) = \frac{(\kappa z)^2}{1 + (\kappa z)^2}. \tag{2.25}$$

Now the task is to rewrite the metric in these coordinates. We already know $f(r)$. We shall let $f(r_0)$ stand as a constant. We also need dr. Take the exterior derivative of the defining equation for the coordinate transformation. It yields

$$dr = \kappa z dz. \tag{2.26}$$

Substituting Eqs. (2.25) and (2.26) on the metric given by Eq. (2.22) we get

$$ds^2 = -\frac{1}{f(r_0)}\frac{(\kappa z)^2}{1+(\kappa z)^2}d\tau_0^2 + [1+(\kappa z)^2]dz^2 + 4M^2\left(1+(\kappa z)^2\right)^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.27}$$

Next we wish to approximate the metric of Eq. (2.27) to lowest order in z around $z = 0$, which characterizes the horizon. The coefficients of dz^2 and $d\theta^2, d\phi^2$ can be seen from inspection. For the $d\tau_0^2$ term, we recall that

$$\frac{(\kappa z)^2}{1+(\kappa z)^2} = (\kappa z)^2 \sum_{n=0}^{\infty}(\kappa z)^{2n}, \tag{2.28}$$

so that to lowest order this is $(\kappa z)^2$ indeed. Thus the approximate metric reads

$$ds^2 \approx -\frac{1}{f(r_0)}(\kappa z)^2 d\tau_0^2 + dz^2 + 4M^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.29}$$

To further identify this near-horizon approximation, we follow [35]. We shall consider a small angular region around the plane $\theta = 0$. In that case, we expand $\sin^2\theta \approx \theta^2$ to lowest order, and we have

$$ds^2 \approx -\frac{1}{f(r_0)}(\kappa z)^2 d\tau_0^2 + dz^2 + 4M^2(d\theta^2 + \theta^2 d\phi^2). \tag{2.30}$$

We see that in this approximation, the angular part of the metric is the same as that of a small piece of a plane in polar coordinates with radius $2M\theta$ and with angular variable ϕ. Thus, one introduces cartesian coordinates $x = 2M\theta\cos\phi$ and $y = 2M\theta\sin\phi$ to get

$$ds^2 \approx -\left(\frac{\kappa z}{\sqrt{f(r_0)}}\right)^2 d\tau_0^2 + dz^2 + dx^2 + dy^2. \tag{2.31}$$

Now we set $a = \kappa/\sqrt{f(r_0)}$. The metric acquires the simple form

$$ds^2 = -(az)^2 d\tau_0^2 + dz^2 + dx^2 + dy^2. \tag{2.32}$$

This approximation is valid for distances near the horizon and for small angular regions around the plane $\theta = 0$. This is the Rindler metric in one of its standard forms [35]. A second useful form which we shall use in Chapter 3, when deriving the Unruh effect, is to define coordinates (η, ξ) through

$$z = \frac{1}{a} e^{a\xi}, \quad \tau_0 = \eta. \tag{2.33}$$

In that case notice that

$$dz = e^{a\xi} d\xi \Longrightarrow dz^2 = e^{2a\xi} d\xi^2, \tag{2.34}$$

while we also have

$$z^2 = \frac{1}{a^2} e^{2a\xi}. \tag{2.35}$$

Substituting Eqs. (2.34) and (2.35) on the metric given by Eq. (2.32) we obtain

$$ds^2 = e^{2a\xi}(-d\eta^2 + dz^2) + dx^2 + dy^2. \tag{2.36}$$

Now one wishes to give a physical interpretation to the parameter a that has been derived. In order to do that, we study the proper acceleration of a Schwarzschild observer. Since the four-velocity of a Schwarzschild observer is just

$$v = f(r)^{-1/2} \partial_t, \tag{2.37}$$

we get the four-acceleration covariantly differentiating it along itself

$$\nabla_v v = f(r)^{-1/2}\nabla_{\partial_t} f(r)^{-1/2}\partial_t = f(r)^{-1}\Gamma^{\mu}_{tt}\partial_{\mu}. \tag{2.38}$$

The connexion coefficients can be obtained easily from the geodesic equation. In particular the only non-vanishing Γ^{μ}_{tt} is Γ^{r}_{tt} and it is given by

$$\Gamma^{r}_{tt} = \frac{M}{r^2} f(r). \tag{2.39}$$

This gives the four-acceleration of a Schwarzschild observer,

$$\nabla_v v = \frac{M}{r^2}\partial_r. \tag{2.40}$$

It follows from this that the magnitude of the acceleration is

$$g(\nabla_v v, \nabla_v v) = \frac{M^2}{r^4} f(r)^{-1}. \tag{2.41}$$

By taking the square root we see that the magnitude of the proper acceleration of a Schwarzschild observer is just

$$\sqrt{g(\nabla_v v, \nabla_v v)} = \frac{1}{\sqrt{f(r)}}\frac{M}{r^2}. \tag{2.42}$$

It can be rewritten entirely in terms of f by noticing that

$$\frac{2M}{r} = 1 - f(r) \Longrightarrow \frac{4M^2}{r^2} = (1 - f(r))^2. \tag{2.43}$$

Using Eq. (2.43) in Eq. (2.42) we can eliminate r^2 from the denominator to get

$$\sqrt{g(\nabla_v v, \nabla_v v)} = \frac{1}{4M\sqrt{f(r)}}(1-f(r))^2 = \frac{\kappa}{\sqrt{f(r)}}(1-f(r))^2. \tag{2.44}$$

This is the form presented in [22] derived by a distinct but equivalent way. One finishes the derivation by relating $1-f(r)$ and κz so that we have

$$1 - f(r) = \frac{1}{1+(\kappa z)^2}. \tag{2.45}$$

Using this information and expanding $\sqrt{g(\nabla_v v, \nabla_v v)}$ to the lowest order in κz one finds out that the acceleration becomes simply

$$\sqrt{g(\nabla_v v, \nabla_v v)} \approx \frac{\kappa}{\sqrt{f(r)}}. \tag{2.46}$$

But the right hand side of the above equation is *exactly* the parameter a we have defined. In that sense, in the near-horizon approximation the parameter a corresponds to the proper acceleration of the Schwarzschild observer at r_0 we have considered in performing the approximation.

Remarkably, the metric given by Eq. (2.32) that encodes the near-horizon geometry of Schwarzschild spacetime has the same functional form of the metric describing the geometry experienced by one observer which accelerates uniformy eternally in Minkowski spacetime. We shall briefly review this, referring the reader to [7] for the full details.

In fact, consider Minkowski spacetime with global coordinates (t, x, y, z) so that the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.47}$$

Now focus on the (t, z) plane, simply by considering that x, y are held constant, and consider the regions $z > |t|$ and $z < -|t|$. They are called respectively the right Rindler wedge and left Rindler wedge and shown as the shaded regions in Fig. (2.2). In the right Rindler wedge, introduce the coordinates

$$t = \frac{1}{a}e^{a\xi}\sinh(a\eta), \quad z = \frac{1}{a}e^{a\xi}\cosh(a\eta). \tag{2.48}$$

The coordinates (η, ξ) are called *Rindler coordinates*. The coordinate η is timelike while

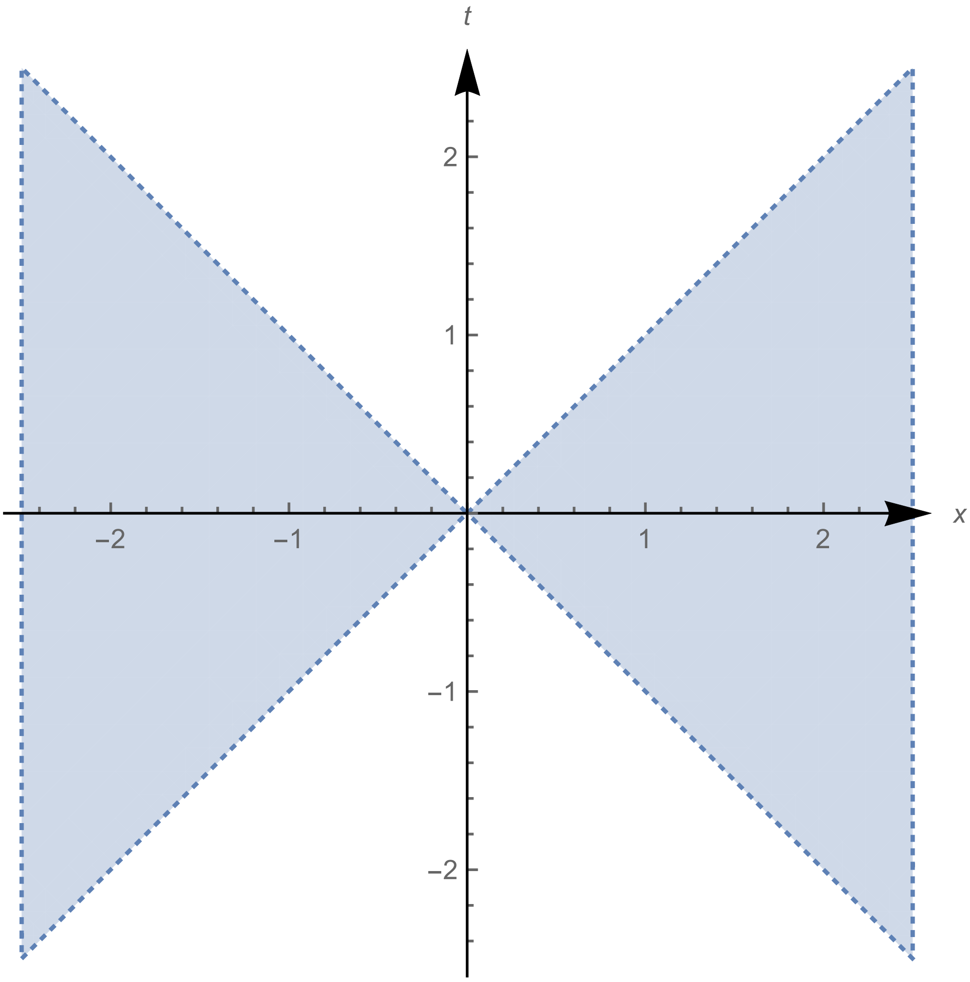

FIGURE 2.2: Diagram of the Rindler Wedges
A diagram of the two Rindler wedges inside Minkowski spacetime. They are shown as the two shaded regions.

the coordinate ξ is spacelike. In these coordinates, the trajectory of an observer which moves with constant acceleration α forever in the positive z direction is parameterized by propertime τ as

$$\eta(\tau) = \frac{\alpha}{a}\tau, \xi(\tau) = \frac{1}{a}\ln\left(\frac{a}{\alpha}\right). \tag{2.49a}$$

$$\tag{2.49b}$$

Thus the lines of constant ξ are the lines describing the motion of these observers and the η is proportional to the propertime along these trajectories. In particular, the line $\xi = 0$ corresponds to an observer moving with acceleration α and η registers its propertime when evaluated along that trajectory. Such an observer is called *a Rindler observer*.

The features discussed above shows these coordinates are adapted to uniformly accelerated motion. In terms of these coordinates, the Minkowski metric restricted to the right Rindler wedge can be written as

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) + dy^2 + dz^2. \tag{2.50}$$

Remarkably, this has exactly the same functional form of the near-horizon metric of Schwarzschild spacetime, expanded to lowest order near the horizon, on the right exterior region, given by Eq. (2.36). In that sense, the right exterior region of Schwarzschild spacetime in the vicinity of the horizon, has the same geometry of the right Rindler wedge in Minkowski spacetime. Moreover, it is clear by the above discussions that the Rindler observer corresponds to a Schwarzschild observer. One may also argue that a Minkowski observer corresponds to a free-falling observer. In fact, a Minkowski observer is in geodesic motion in Minkowski spacetime, and in particular, in the right Rindler wedge. Now the map from the right Rindler wedge to the near-horizon region of Schwarzschild spacetime is an isometry, so it is geodesic-preserving. In that sense, the Minkowski observer gets mapped to a free-falling observer in geodesic motion near the horizon of the Schwarzschild geometry.

Finally we consider the left Rindler wedge, which is the analogue of the left Schwarzschild exterior region. Analogously to what has been done in the right Rindler wedge, one may define coordinates

$$t = -\frac{1}{a}e^{a\xi}\sinh(a\eta), \quad z = -\frac{1}{a}e^{a\xi}\cosh(a\eta). \tag{2.51}$$

The reason to use the same notation (η, ξ), for these coordinates on the left Rindler wedge, that we used for the coordinates on the right Rindler wedge, is that the metric will acquire the same functional form in both of them [7].

3

Quantum Mechanics and Correlations

In this chapter our aim is to review standard material on quantum mechanics. The objective is to establish the notation and terminology to be used afterwards and to review some results. We begin with the idea of mixed states, composite systems and entanglement. Finally we talk about correlations and information, which are standard in quantum information theory.

3.1 Mixed States and Uncertainty

The standard postulates of quantum mechanics (reviewed in the notation we use here in Appendix D) can be generalized by introducing the notion of a mixed state, which is the aim of this section. This material can be found, *e.g.* in [42], however we adapt to our notations which are reviewed in Appendix D. The simplest way to generalize this is to consider that for some reason there is uncertainty in the microstate of the system. This is a well-known line of thought from Statistical Mechanics. We shall revisit now the probability and mean value formulas for this case, but we shall work in the language of spectral measures because of their generality, which cover both discrete and continuous spectra. In that case, suppose that one knows somehow that there is an ensemble $\{(p_i, |\psi_i\rangle)\}$ of states with probability distribution p_i. This means that there is probability p_i that the system is described by the state $|\psi_i\rangle$. Let A be one observable with spectrum $\sigma(A)$ and $\mathbb{P}_A$ the associated projection-valued measure. What is the probability that a measurement of A lies in $S \subset \sigma(A)$ when one is assured the state of the system is $|\psi_i\rangle$? By the postulates of Quantum Mechanics it is

$$P(S|i) = \langle\psi_i|\mathbb{P}_A(S)|\psi_i\rangle. \tag{3.1}$$

Considering the uncertainty in the state, by the usual rules of probability theory, the probability for a measurement of A to give a result inside S is

$$P(S) = \sum_i p_i P(S|i) = \sum_i p_i \langle\psi_i|\mathbb{P}_A(S)|\psi_i\rangle. \tag{3.2}$$

To further make progress, recall that if $E_1 \cap E_2 = \emptyset$ then $\mathbb{P}_A(E_1)\mathscr{H}$ and $\mathbb{P}_A(E_2)\mathscr{H}$ are orthogonal subspaces. This holds true for any $S \subset \sigma(A)$ and its complement $\tilde{S} = \sigma(A) \setminus S$. Furthermore, since $\mathbb{P}_A(\sigma(A)) = \mathbf{1}$ is the identity it holds that

$$\mathbb{P}_A(S) + \mathbb{P}_A(\tilde{S}) = \mathbf{1}, \tag{3.3}$$

so that $\mathbb{P}_A(\tilde{S})$ is the complementary projector. This implies the decomposition of the Hilbert space as an orthogonal direct sum

$$\mathscr{H} = \mathbb{P}_A(S)\mathscr{H} \oplus \mathbb{P}_A(\tilde{S})\mathscr{H}. \tag{3.4}$$

Therefore we can pick an orthonormal basis $\{|\psi_m\rangle : m \in I \subset \mathbb{N}\}$ of $\mathscr{H}$ such that for $m \in I_S \subset I$ the $\{|\psi_m\rangle\}$ form an orthonormal basis of $\mathbb{P}_A(S)\mathscr{H}$. Now, any $|\psi\rangle$ can be written as

$$
\begin{aligned}
|\psi\rangle &= \sum_{m\in I}\langle\psi_m|\psi\rangle|\psi_m\rangle \\
&= \sum_{m\in I_S}\langle\psi_m|\psi\rangle|\psi_m\rangle + \sum_{m\notin I_S}\langle\psi_m|\psi\rangle|\psi_m\rangle,
\end{aligned} \tag{3.5}
$$

and by applying $\mathbb{P}_A(S)$ we get

$$
\mathbb{P}_A(S)|\psi\rangle = \sum_{m\in I_S}\langle\psi_m|\psi\rangle|\psi_m\rangle. \tag{3.6}
$$

This means that, by the arbitrariness of $|\psi\rangle$, the projector $\mathbb{P}_A(S)$ can be written concretely as

$$
\mathbb{P}_A(S) = \sum_{m\in I_S}|\psi_m\rangle\langle\psi_m|. \tag{3.7}
$$

In turn, taking $\{|\phi_m\rangle \in \mathscr{H} : m \in I_S\}$ as an orthonormal basis of $\mathbb{P}_A(S)\mathscr{H}$ indexed by I_S, it gives the desired probability as

$$
\begin{aligned}
P(S) &= \sum_i p_i P(S|\Psi_i) \\
&= \sum_i p_i\langle\psi_i|\mathbb{P}_A(S)|\psi_i\rangle \\
&= \sum_i\sum_{m\in I_S} p_i\langle\psi_i|\phi_m\rangle\langle\phi_m|\psi_i\rangle \\
&= \sum_{m\in I_S}\langle\phi_m|\sum_i p_i|\psi_i\rangle\langle\psi_i|\phi_m\rangle.
\end{aligned} \tag{3.8}
$$

This suggests defining the operator ρ by

$$
\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \tag{3.9}
$$

from which it follows that

$$P(S) = \sum_{m \in I_S} \langle \phi_m | \rho | \phi_m \rangle. \tag{3.10}$$

Now recall that $\mathbb{P}_A(S)|\phi_m\rangle = |\phi_m\rangle$ when $m \in I_S$ and zero otherwise. Hence we can let the sum run over all the basis changing $|\phi_m\rangle$ by $\mathbb{P}_A(S)|\phi_m\rangle$ and we get

$$P(S) = \sum_{m \in I} \langle \phi_m | \mathbb{P}_A(S)^\dagger \rho \mathbb{P}_A(S) | \phi_m \rangle. \tag{3.11}$$

Finally we recognize this as a trace

$$P(S) = \mathrm{Tr}(\mathbb{P}_A(S)^\dagger \rho \mathbb{P}_A(S)), \tag{3.12}$$

which upon recalling that $\mathbb{P}_A(S)$ is a hermitian projector and using the cyclic property of the trace yields

$$P(S) = \mathrm{Tr}(\mathbb{P}_A(S)\rho). \tag{3.13}$$

The mean value now follows immediately from a simple observation from measure theory based on the above result. Indeed let $f : \sigma(A) \to \mathbb{C}$ be a function defined on the spectrum of the observable. Suppose it is integrable against the probability measure P and the projection valued measure $\mathbb{P}_A$. Then it holds

$$\int_{\sigma(A)} f(\lambda) dP(\lambda) = \mathrm{Tr}\left[\int_{\sigma(A)} f(\lambda) d\mathbb{P}_A(\lambda) \rho\right]. \tag{3.14}$$

In that case, recalling that the mean-value is

$$\langle A \rangle = \int_{\sigma(A)} \lambda dP(\lambda), \tag{3.15}$$

we get

$$\langle A \rangle = \mathrm{Tr}\left[\int_{\sigma(A)} \lambda d\mathbb{P}_A(\lambda)\rho\right] = \mathrm{Tr}\, A\rho, \tag{3.16}$$

where in the last step we have invoked the spectral decomposition of A.

So far we have seen that one classical ensemble of quantum states can be encoded in an operator of the form

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \tag{3.17}$$

and furthermore, we have seen that the probability measure of any observable can be completely characterized in terms of ρ and the spectral measure.

Finally we should discuss what happens to the collapse of the state after the measurement has been carried out. Indeed, suppose the state of the system were $|\psi_i\rangle$. Then by the postulates, if the measurement result lies in $S \subset \sigma(A)$ then the post-selected state is

$$|\psi_i'\rangle = \frac{\mathbb{P}_A(S)|\psi_i\rangle}{\sqrt{P(S|i)}}. \tag{3.18}$$

Since the state $|\psi_i\rangle$ occurs with probability p_i, in the post-measurement case we get another classical ensemble of quantum states. What are the probabilities for the ensemble? For the post-selected state to be $|\psi_i'\rangle$ the state prior to measurement must be $|\psi_i\rangle$ *and* the measurement result must lie in S. In other words, the probability for this to happen is the normalized product $Np_iP(S|i)$, where the normalization term N is obtained summing over i and equating to total probability 1. It turns out that we already computed the sum that defines the normalization. It equals $P(S) = \mathrm{Tr}\,\mathbb{P}_A(S)\rho$. In that case, we finally assemble the density operator for the post-selected ensemble

$\{(p'_i, |\psi'_i\rangle)\}$

$$\begin{aligned}\rho' &= \sum_i p'_i |\psi'_i\rangle\langle\psi'_i| \\ &= \sum_i \frac{p_i P(S|i)}{P(S)} \frac{\mathbb{P}_A(S)|\psi_i\rangle}{\sqrt{P(S|i)}} \frac{\langle\psi_i|\mathbb{P}_A(S)^\dagger}{\sqrt{P(S|i)}} \\ &= \frac{1}{P(S)} \sum_i p_i \mathbb{P}_A(S)|\psi_i\rangle\langle\psi_i|\mathbb{P}_A(S)^\dagger \\ &= \frac{1}{P(S)} \mathbb{P}_A(S)\rho\mathbb{P}_A(S)^\dagger, \end{aligned} \tag{3.19}$$

now using the fact that $\mathbb{P}_A(S)$ is a Hermitian projector we can further simplify this to

$$\rho' = \frac{1}{\operatorname{Tr} \mathbb{P}_A(S)\rho} \mathbb{P}_A(S)\rho\mathbb{P}_A(S). \tag{3.20}$$

So, we are able to directly compute in terms of the density operator and the spectral measure the probability measure for the observable, its mean value and the post-selected state. The density operator enjoys two main properties. First of all, it is Hermitian. This is obvious from the way it was defined. Second, it has unit trace. This follows from the fact that for any observable A, the normalization of the probability and spectral measures P and $\mathbb{P}_A$ are related by

$$\int_{\sigma(A)} dP(\lambda) = \operatorname{Tr}\left[\int_{\sigma(A)} d\mathbb{P}_A(\lambda)\rho\right]. \tag{3.21}$$

Inserting the normalization conditions, satisfied by all observables, it holds that

$$\operatorname{Tr}\rho = 1. \tag{3.22}$$

There is an important point to understand here. we have started with a classical ensemble of quantum states, then we found out that all probability measures for observables and hence all mean values, are characterized entirely in terms of traces involving the operator ρ. It turns out that two classical ensembles $\{(p_i, |\psi\rangle)\}$ and $\{(p'_i, |\psi'_i\rangle)\}$ may give rise to the same ρ. To make this clear, consider a two-level quantum state whose Hilbert space is spanned by a basis of two states $|0\rangle, |1\rangle$. Such is the case of the description of the isolated spin degree of freedom of one electron. As we shall further discuss in the next section, the states of two such particles is described on the tensor product of two copies of this Hilbert space whose basis is $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

Consider now the ensemble

$$\left\{ \left(\frac{1}{2}, |00\rangle \right), \left(\frac{1}{2}, |11\rangle \right) \right\}. \tag{3.23}$$

Its corresponding density operator is the simple one

$$\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|. \tag{3.24}$$

Next consider the ensemble

$$\left\{ \left(\frac{1}{2}, \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \right), \left(\frac{1}{2}, \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \right) \right\}. \tag{3.25}$$

Its corresponding density operator is remarkably the same as the previous one in Eq. (3.24). Therefore all predictions of quantum mechanics, namely mean values and probabilities, are the same in both ensembles. In other words, they are *indistinguishable* ensembles and should be deemed equivalent. In that case, since all quantities of theoretical interest are contained in such operator and are actually independent of the particular expression in terms of some ensemble of quantum states, we generalize the notion of state by saying that a quantum state is just a Hermitian, unit-trace operator, called the density operator. The unit ray quantum state is a particular case of this. Such a state is called pure, because it does not involve any classical uncertainty on its specification. When the density operator is not pure we call it mixed. In these terms the postulates of quantum mechanics are revisited to become:

- The states of a system are hermitian, unit-trace operators acting on a separable Hilbert space $\mathscr{H}$, called density operators. The state is called pure when it is the projector onto a ray, otherwise it is called mixed.
- The quantities one can measure associated to a system are described by hermitian operators on the system's Hilbert space. These operators are called observables.
- The possible values a physical quantity may attain are the elements of the spectrum of the observable.
- Let A be an observable with projective measure $\mathbb{P}_A$ defined on the Borel sigma algebra of its spectrum $\sigma(A)$ by means of the spectral theorem. If the state of the system is ρ the probability that a measurement of A lies in $S \subset \sigma(A)$ is

$$P(S) = \mathrm{Tr}\, \mathbb{P}_A(S)\rho. \tag{3.26}$$

- In the conditions of the previous postulate, if A is measured in state ρ and the result lies in $S \subset \sigma(A)$ then the post-selected state after the measurement is the normalized projection

$$\rho' = \frac{1}{P(S)}\mathbb{P}_A(S)\rho\mathbb{P}_A(S). \tag{3.27}$$

Next we would like to quantify the uncertainty in the quantum state encoded in ρ. As justified already, there are many descriptions of ρ in terms of pure states. Nevertheless there is one which is quite natural. Since ρ is hermitian, we can form one ensemble

$\{(p_i, |\psi_i\rangle)\}$ where p_i are its eigenvalues and $|\psi_i\rangle$ its eigenvectors. In this basis the operator is diagonal

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{3.28}$$

Since the random nature of the state is encoded in the probability distribution p_i for the various pure states $|\psi_i\rangle$ possible, the uncertainty is about this probability distribution. In turn, it is well-known that the uncertainty can be quantified by the Shannon entropy,

$$S(\{p_i\}) = -\sum_i p_i \log p_i \tag{3.29}$$

The entropy can be written in terms of ρ itself, by noticing that it is actually

$$S(\rho) = -\operatorname{Tr} \rho \log \rho, \tag{3.30}$$

which is the Von-Neumman entropy, and the log is usually understood in basis 2, even though in Statistical Mechanics it is common to work with the natural logarithm. We can further show that ρ is pure if and only if $S(\rho) = 0$. Indeed, in the way it was defined, $S(\rho)$ equals the Shannon entropy of the probability distribution p_i. It is a simple result in probability theory that $S(\{p_i\})$ is zero if and only if all p_i are zero except one which is unit. When that happens, by the ensemble decomposition of ρ, it means $\rho = |\psi_i\rangle\langle\psi_i|$ where p_i is the non-zero value. Thus $S(\rho) = 0$ if and only if ρ is pure. It is interesting to notice that the entropy yields a measure of purity. To do so, we shall expand Eq. (3.30). Recall that, to first order in x the log in an arbitrary basis reads

$$\log_b x \approx \frac{1}{\ln b}(x - 1), \tag{3.31}$$

hence we have

$$\begin{aligned} S(\rho) &\approx -\operatorname{Tr}\rho \frac{1}{\ln b}(\rho - \mathbf{1}) \\ &= -\frac{1}{\ln b}[\operatorname{Tr}\rho^2 - \operatorname{Tr}\rho] \\ &= -\frac{1}{\ln b}[\operatorname{Tr}\rho^2 - 1] \\ &= \frac{1}{\ln b}[1 - \operatorname{Tr}\rho^2]. \end{aligned} \tag{3.32}$$

Regardless of the basis of the log, we see that when the eigenvalues of ρ are sufficiently small, it holds true that $S(\rho)$ is completely characterized by $\operatorname{Tr}\rho^2$. Further $S(\rho)$ seems to be complementary to $\operatorname{Tr}\rho^2$ since as one grows the other must diminish. This is the motivation to define this last quantity as the purity of the state.

Definition 3.1.1. Let a quantum system be given with state ρ acting on the Hilbert space $\mathscr{H}$. We define the purity of the state to be $\operatorname{Tr}\rho^2$.

We can show independently of the entropy that a state is pure if and only if $\operatorname{Tr}\rho^2 = 1$. Indeed if it is pure, $\rho^2 = \rho$, since there is a $|\psi\rangle$ such that $\rho = |\psi\rangle\langle\psi|$, which makes the property follow immediately. If $\operatorname{Tr}\rho^2 = 1$, writing this equation using the spectral decomposition of ρ it follows that $\sum_i p_i^2 = 1$. On the other hand

$$\left(\sum_i p_i\right)^2 = \sum_i p_i^2 + 2\sum_{i<j} p_i p_j, \tag{3.33}$$

which implies under the assumed conditions that

$$\sum_{i<j} p_i p_j = 0. \tag{3.34}$$

Eq. (3.34) is a sum of positive numbers, which is zero if and only if all numbers are zero. The only way for this to hold with $\sum_i p_i = 1$ is that all except one p_i is zero, which is when ρ is pure. Thus ρ is indeed pure if and only if the purity, $\operatorname{Tr}\rho^2$, is one.

These short remarks on mixed states are already enough for us to introduce the idea of quantum entanglement.

3.2 Composite Systems and Entanglement

Now we turn to the description of composite systems, on which we shall see the density operators play a fundamental role. In the usual approach this is dealt using the tensor product Hilbert space. It is important to notice, however, that there is no absolute division of a system into parts. This kind of description usually occurs for the reason that each part of the system is observed by a distinct observer. This is many times associated to a real spatial separation between the two parts. The important aspect is that most of the times, some observer is *limited* to observe just one part of the system.

We shall first define what we mean by local operators in this case.

Definition 3.2.1. Let a quantum system be given whose Hilbert space decomposes as $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$. The operators of the form $E \otimes \mathbf{1}$ where E is any linear operator in $\mathscr{H}_A$ is called *local* to A, and analogously for operators of the form $\mathbf{1} \otimes E$ where E is any linear operator in $\mathscr{H}_B$.

In this decomposition, the observables of the system which can actually be measured by one observer restricted to either part are the corresponding local observables. It is intuitive that if the observer restricted to one part of the system simply has no access to the other, he would describe a quantum theory from his point of view without any mention to the other part. So one natural question arises: how does that description comes by related to a global description that some external observer could have? To understand that we must understand that an observer which just observes part A will probe local observables to A as in Def. 3.2.1. The associated probability measures and mean values all come from traces. So we better understand how the trace works in tensor products compared to its working on the individual parts. This all starts with the following proposition, which is in fact easy to prove:

Proposition 3.2.1. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ and let A, B be operators respecitvely on $\mathscr{H}_A$ and $\mathscr{H}_B$, then

$$\operatorname{Tr} A \otimes B = (\operatorname{Tr} A)(\operatorname{Tr} B). \tag{3.35}$$

Proof. Let $|\psi_A^n\rangle, |\psi_B^m\rangle$ be bases of $\mathscr{H}_A$ and $\mathscr{H}_B$. Then $|\psi_A^n\rangle \otimes |\psi_B^m\rangle$ is a basis of $\mathscr{H}$ indexed by pairs (n, m). Then by definition of the trace

$$\begin{aligned}\operatorname{Tr} A \otimes B &= \sum_{n,m} \langle \psi_A^n | \otimes \langle \psi_B^m | A \otimes B | \psi_A^n \rangle \otimes \psi_B^m \rangle \\ &= \sum_{n,m} \langle \psi_A^n | A | \psi_A^n \rangle \langle \psi_B^m | B | \psi_B^m \rangle \\ &= (\operatorname{Tr} A)(\operatorname{Tr} B),\end{aligned} \tag{3.36}$$

and the proposition is proved. □

Now, of course not all operators are tensor products, but it is clearly true that *all operators are generated by tensor products*. This is simple to see considering the basis projectors $|\psi_A^n\rangle\langle\psi_A^m| \otimes |\psi_B^k\rangle\langle\psi_B^l|$ which form a basis for all operators. In that case, we always have, for an arbitrary operator O the decomposition:

$$O = \sum_{nm} O_{nm} A_n \otimes B_m. \tag{3.37}$$

In that case, by linearity of the trace we immediately obtain

$$\operatorname{Tr} O = \sum_{nm} O_{nm} \operatorname{Tr}(A_n) \operatorname{Tr}(B_m). \tag{3.38}$$

For example, using the basis projectors as we have mentioned briefly above, we would decompose

$$O = \sum_{nmkl} O_{nkml} |\psi_A^n\rangle\langle\psi_A^m| \otimes |\psi_B^k\rangle\langle\psi_B^l|, \tag{3.39}$$

and it follows that

$$\begin{aligned} \operatorname{Tr} O &= \sum_{nmkl} O_{nkml} \operatorname{Tr}(|\psi_A^n\rangle\langle\psi_A^m|) \operatorname{Tr}(|\psi_B^k\rangle\langle\psi_B^l|) \\ &= \sum_{nmkl} O_{nkml}\delta_{nm}\delta_{kl} \\ &= \sum_{nm} O_{nmnm}, \end{aligned} \tag{3.40}$$

which is nothing but the usual definition of the trace.

Armed with these ideas, let ρ be a global bipartite state, which can be mixed, and let $E \otimes \mathbf{1}$ be a local operator on the first factor. We wish to compute $\operatorname{Tr} \rho(E \otimes \mathbf{1})$. For that decompose

$$\rho = \sum_{nm} \rho_{nm} A_n \otimes B_m, \tag{3.41}$$

thus we immediately get

$$\rho(E \otimes \mathbf{1}) = \sum_{nm} \rho_{nm} (E A_n) \otimes B_m, \tag{3.42}$$

consequently, taking the trace it follows that

$$\operatorname{Tr} \rho(E \otimes \mathbf{1}) = \sum_{nm} \rho_{nm} \operatorname{Tr}(E A_n) \operatorname{Tr} B_m. \tag{3.43}$$

But now notice Eq. (3.43) can be written as

$$\begin{aligned}\operatorname{Tr}\rho(E\otimes\mathbf{1}) &= \operatorname{Tr}\left[\left(\sum_n(\sum_m \operatorname{Tr} B_m\rho_{nm})A_n\right)E\right] \\ &= \operatorname{Tr}\rho_A E,\end{aligned} \tag{3.44}$$

where we have defined

$$\rho_A = \sum_n\sum_m \operatorname{Tr} B_m\rho_{nm}A_n. \tag{3.45}$$

It is easy to see that ρ_A is a state in $\mathscr{H}_A$. The operation we have just carried out is the so-called partial trace: we have averaged over B to get the state actually observed by A.

To make this operation more systematic, we can define it in the following way:

Definition 3.2.2. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$. Let $\mathfrak{L}(\mathscr{H})$ be the space of linear operators on $\mathscr{H}$ and $\mathfrak{L}(\mathscr{H}_A)$ the corresponding space of linear operators on $\mathscr{H}_A$. We define the partial trace over B to be the mapping $\operatorname{Tr}_B : \mathfrak{L}(\mathscr{H}) \to \mathfrak{L}(\mathscr{H}_A)$ defined on decomposable operators as

$$\operatorname{Tr}_B(A\otimes B) = (\operatorname{Tr} B)A, \tag{3.46}$$

and extended by linearity and continuity to the other operators. We define $\operatorname{Tr}_A : \mathfrak{L}(\mathscr{H}) \to \mathfrak{L}(\mathscr{H}_B)$ in the same way.

Before proceeding, we show a simple property of the partial trace that will be used in Chapter 5: it satisfies the same cyclic property of the usual trace with respect to multiplication by local operators. Making this more precise we prove the following proposition

Proposition 3.2.2. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$. Let $\mathcal{O}$ an arbitrary operator in $\mathscr{H}$ and $E \otimes \mathbf{1}$ one local operator. Then

$$\mathrm{Tr}_A((\mathbf{E} \otimes \mathbf{1})\mathcal{O}) = \mathrm{Tr}_A(\mathcal{O}(\mathbf{E} \otimes \mathbf{1})). \tag{3.47}$$

Proof. First we prove for the case when $\mathcal{O}$ is a product operator $\mathcal{O} = A \otimes B$. In that case

$$(E \otimes \mathbf{1})\mathcal{O} = (EA) \otimes B. \tag{3.48}$$

Taking the partial trace over A using the previous definition gives

$$\begin{aligned}
\mathrm{Tr}_A\left[(E \otimes \mathbf{1})\mathcal{O}\right] &= \mathrm{Tr}_A\left[(E \otimes \mathbf{1})(A \otimes B)\right] \\
&= \mathrm{Tr}_A\left[EA \otimes B\right] \\
&= \mathrm{Tr}(EA)B.
\end{aligned}$$

But the above trace is the usual scalar-valued trace and hence $\mathrm{Tr}(EA) = \mathrm{Tr}(AE)$, and this gives by reverting the above steps

$$\begin{aligned}
\mathrm{Tr}_A\left[(E \otimes \mathbf{1})\mathcal{O}\right] &= \mathrm{Tr}(AE)B \\
&= \mathrm{Tr}_A\left[AE \otimes B\right] \\
&= \mathrm{Tr}_A\left[(A \otimes B)(E \otimes \mathbf{1})\right] \\
&= \mathrm{Tr}_A\left[\mathcal{O}(E \otimes \mathbf{1})\right]
\end{aligned} \tag{3.49}$$

and the property holds. For a general operator $\mathcal{O}$ the result follows by linearity. □

To summarize, this discussion shows the following: if ρ is a quantum state on a Hilbert space admitting a bipartition $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ and $E \otimes \mathbf{1}$ one local operator to A, then $\operatorname{Tr}(E \otimes \mathbf{1})\rho$ can be computed as a trace in $\mathscr{H}_A$ with the operator E and ρ replaced by ρ_A.

For an observer which has access just to part A of the system, or simply does not care about part B, the observables of interest are exactly the local observables to A. That means that all the probability measures and mean values of interest for this observer can be written without any mention to B by working entirely with $\mathscr{H}_A$ and considering the state to be ρ_A.

The important result we have derived is that if ρ is the quantum state of a composite system with Hilbert space $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$, then for observers probing only the A or B subsystems, the state of the effective system they observe is the reduced density operator $\rho_A = \operatorname{Tr}_B \rho$ or $\rho_B = \operatorname{Tr}_A \rho$. Remarkably, even if ρ is pure, it might be the case that ρ_A or ρ_B are not. In plain English: even if one knows without any uncertainty the state of a quantum system, it is possible that the states of its subsystems are unknown and carry uncertainty. That is what we call *quantum entanglement*. So we start with this provisional definition:

Definition 3.2.3. Let a quantum system be given with Hilbert space $\mathscr{H}$. Suppose the state of the system at some instant is $|\psi\rangle$. Suppose further there is a bipartition $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$. If the partial states $\rho_A = \operatorname{Tr}_B |\psi\rangle\langle\psi|$ and $\rho_B = \operatorname{Tr}_A |\psi\rangle\langle\psi|$ are mixed, then we say that $|\psi\rangle$ is entangled. Otherwise we say that $|\psi\rangle$ is unentangled.

Interestingly Def. (3.2.3), indirectly, states a measure of entanglement: the von-Neumann entropies $S(\rho_A)$ and $S(\rho_B)$ of the partial states. This is a measure of the mixedness of the partial states and is zero if and only if they are pure. In that case, they are zero if and only if $|\psi\rangle$ is unentangled. Nevertheless, it is possible to show that $S(\rho_A) = S(\rho_B)$. For that we recall a result in Functional Analysis, which we will not prove, regarding the tensor product of Hilbert spaces. That is the so-called Schmidt decomposition:

Theorem 3.2.1. Let $\mathscr{H}_A$ and $\mathscr{H}_B$ be Hilbert spaces and let $|\psi\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$. Then there exists orthonormal bases $|\phi_n^A\rangle$ and $|\phi_m^B\rangle$ of $\mathscr{H}_A$ and $\mathscr{H}_B$ respectively and scalars $\lambda_n \in \mathbb{C}$ such that

$$|\psi\rangle = \sum_n c_n |\phi_n^A, \phi_n^B\rangle. \tag{3.50}$$

This result is useful because it furnishes a simple expression for the reduced density operators. Indeed with this result if follows that

$$\rho = |\psi\rangle\langle\psi| = \sum_{nm} c_n c_m^* |\phi_n^A, \phi_n^B\rangle\langle\phi_m^A, \phi_m^B|. \tag{3.51}$$

The partial state ρ_A is

$$\begin{aligned}\rho_A &= \operatorname{Tr}_B \rho \\ &= \sum_n |c_n|^2 |\phi_n^A\rangle\langle\phi_n^A|,\end{aligned} \tag{3.52}$$

and the exactly same holds true for ρ_B,

$$\begin{aligned}\rho_B &= \operatorname{Tr}_A \rho \\ &= \sum_n |c_n|^2 |\phi_n^B\rangle\langle\phi_n^B|.\end{aligned} \tag{3.53}$$

This in turn shows that $S(\rho_A) = S(\rho_B)$ and that both are equal to

$$S(\rho_A) = S(\rho_B) = -\sum_n |c_n|^2 \log |c_n|^2, \tag{3.54}$$

where c_n are the Schmidt decomposition coefficients of $|\psi\rangle$.

One immediate result follows: this common entropy is zero if and only if just one $|c_n| = 1$ and all the other are zero. But that is the case on which $|\psi\rangle$ is a tensor product state. So what we have established is that:

Theorem 3.2.2. Let a quantum system be given and let $\mathscr{H}$ be its Hilbert space with $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$. Suppose the system is in the state $|\psi\rangle$. Then $|\psi\rangle$ is entangled if and only if $|\psi\rangle$ cannot be written as a tensor product, or equivalently, $|\psi\rangle$ is unentangled if and only if it is a tensor product state.

Therefore, we see from the above discussion that entanglement is actually related to separability of a state. A state is entangled if and only if it is not separable, where separability means that the state can be written as a product.

So far we have dealt with global pure states. We generalize to global mixed states by means of the following definition:

Definition 3.2.4. Let a quantum system be given with Hilbert space $\mathscr{H}$ with bipartition $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$. Then a quantum state ρ is said to be separable when it can be written as

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i, \tag{3.55}$$

for an arbitrary ensemble of states $\{(p_i, \rho_A^i, \rho_B^i)\}$. When the state is not separable it is said to be entangled.

This definition reduces to the previous one in the case of a global pure state. Nevertheless, detecting and quantifying entanglement of a mixed state is a much harder task than for pure states.

3.3 Measures of Entanglement

We have dealt with the question: is one state entangled or not? For pure states the question is equivalent to whether or not the partial states are mixed and the von-Neumman entropy allows not just to answer this question, but rather to provide a *measure of entanglement*. For mixed states the matter is more complicated as the von-Neumman entropy of the partial states by itself is not suitable anymore [37].

So first we should clarify what a measure of entanglement for a state ρ_{AB} on $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ should be. A minimum set of assumptions on what an entanglement measure should satisfy is [4, 17]:

1. $E(\rho_{AB})$ should be symmetric under the interchange of systems A and B.
2. $E(\rho_{AB})$ must be non-negative, and zero if and only if ρ_{AB} is separable;
3. $E(\rho)$ must be monotonic under local operations and classical communication;
4. If $\rho = \sum_i p_i \rho^i$ where ρ^i is pure and $\sum p_i = 1$ we have
$$E(\rho) \leq \sum p_i E(\rho^i);$$
5. $E(\rho_{AB})$ should reduce to the entanglement entropy when ρ_{AB} is pure;

It is important to remark that in [17] the authors state the above properties in the language of the algebraic approach to quantum mechanics utilizing $*$-algebras. They also require a continuity property that we omitted for simplicity. The same continuity property is required, in the usual density operator notation, in [4]. This last reference also requires some additional properties, but we wanted here to state just the bare minimum requirements. We also refer the reader to [4] for definitions of local operations and classical communication.

The problem of finding such a measure is highly non-trivial with many of them being suggested in the literature [37]. Even then, if we have a definition of such a measure, it is usually the case that we simply do not know how to compute it analytically. We shall here talk about mainly two measures in the above sense - the relative entropy of entanglement and the entanglement of formation - and a third quantity - the negativity - which, although does not reduce to the Von-Neumman entropy on pure states, has the advantage of being rather easy to compute and being able to at least *detect* entanglement. Still it also does not satisfy the first property: indeed if the negativity is non-zero, one may say that the state is entangled, but even if the negativity is zero the state may be entangled and such entanglement may not be detected. This is, in fact, the reason why the authors in [4] weaken the first requirement to the condition that $E(\rho) = 0$ if ρ is separable without demanding the converse.

So, let us start with the relative entropy of entanglement. This measure is based on one important quantity called the relative entropy. This is defined as follows:

Definition 3.3.1. Let $\mathscr{H}$ be the Hilbert space of a quantum system and let ρ, σ be two states on it. We define the relative entropy, or Kullback-Leibler divergence as

$$S(\rho|\sigma) = -\operatorname{Tr}\rho(\log\rho - \log\sigma). \tag{3.56}$$

The relative entropy is a measure of divergence of two quantum states. Considering ρ to be the actual quantum state of the system, it measures how much information is lost if one uses σ to describe the system. In that case, let ρ_{AB} be the given true state of a system. Let σ be an arbitrary separable state. We can construct one entanglement measure by comparing ρ_{AB} with σ. This is a measure of how much knowledge one looses by taking a separable state instead of ρ_{AB} to describe the physical situation. This gives rise to the following definition of a measure of entanglement:

Definition 3.3.2. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ be a bipartite Hilbert space and let ρ_{AB} be a state. We define the relative entropy of entanglement to be

$$E_R(\rho_{AB}) = \inf_{\sigma_{AB} \in \mathrm{Sep}} S(\rho_{AB}|\sigma_{AB}), \tag{3.57}$$

where Sep is the space of separable states.

$E_R(\rho_{AB})$ has shown to be a true entanglement measure in the above sense in a very general context which involves Quantum Field Theory in [17]. Although we shall not actually use this measure to perform any analysis, given its importance specially in the context of Quantum Field Theory, we find opportune by completeness to include it in this review.

The second measure, which we shall actually see how to compute in one special situation, is the entanglement of formation. This measure has two positive points to it: the first one is that it has a special relation to the classical correlation, a measure of accessible information through measurements that we are going to define, and the second advantage is that it has an operational interpretation that makes it a good measure for applications. The definition goes as follows:

Definition 3.3.3. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ be the Hilbert space of a quantum system and let its state be ρ_{AB}. We define the entanglement of formation of ρ_{AB} to be

$$E_F(\rho_{AB}) = \inf_{\{(|\psi_i\rangle, p_i)\}} \sum_i p_i S(\operatorname{Tr}_A(|\psi_i\rangle\langle\psi_i|)) = \inf_{\{(|\psi_i\rangle, p_i)\}} \sum_i p_i S(\operatorname{Tr}_B(|\psi_i\rangle\langle\psi_i|)), \tag{3.58}$$

where the infimum is taken over the set of all ensembles that realize ρ_{AB}.

In the above definition it is paramount to recall that a density operator has many realizations as "ensembles of pure states" which yields the same probability distributions for all observables. The infimum is taken over these various realizations.

Now that we have stated these two important entanglement measures, notice that both of them involve optimizations on their definitions: they are both defined as the greatest lower bound taken over sets which can be quite complicated to parameterize in practice. Because of that it is very rare that one is able to actually compute any of these analytically. Indeed, there are methods to compute such measures in special cases, as we shall see, but in general these still require one to work numerically. In other cases what one has available are lower and upper bounds. In particular, for the case of the relative entropy of entanglement, in the specific context of Quantum Field Theory, many such bound results are proven in [17].

Finally, we turn to the third measure, which is not actually an entanglement measure in the sense we have defined, but which is able to detect when a state is entangled, with the advantage of being easy to compute in general. This is the so-called negativity. But to define it, we first need the idea of partial transposition:

Definition 3.3.4. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ be a Hilbert space and let $|\psi_n^A\rangle$ and $|\psi_m^B\rangle$ be bases of $\mathscr{H}_A, \mathscr{H}_B$ so that we get a basis of $\mathscr{H}_A \otimes \mathscr{H}_B$ as

$$|\psi_n^A, \psi_m^B\rangle = |\psi_n^A\rangle \otimes |\psi_m^B\rangle. \tag{3.59}$$

The operators $|\psi_n^A, \psi_m^B\rangle\langle\psi_{n'}^A, \psi_{m'}^B|$ form a basis of the space of operators on $\mathscr{H}$. We define the partial transposition operation over B, denoted T_B on such operators by

$$(|\psi_n^A, \psi_m^B\rangle\langle\psi_{n'}^A, \psi_{m'}^B|)^{T_B} = |\psi_n^A, \psi_{m'}^B\rangle\langle\psi_{n'}^A, \psi_m^B|, \tag{3.60}$$

and extend by linearity and continuity to all operators on $\mathscr{H}$.

Definition 3.3.5. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$. Let ρ be a state. We define its negativity with respect to A to be

$$\mathcal{N}_A(\rho) = \frac{\|\rho^{T_A}\|_1 - 1}{2}, \tag{3.61}$$

where

$$\|X\|_1 = \operatorname{Tr} \sqrt{X^\dagger X} \tag{3.62}$$

is called the trace norm of X.

The advantage of negativity is that in general it is easy to compute. In truth all one needs to do is to take the partial transposition over the subsystem of interest and find the eigenvalues of the partial transposed density operator. In that case, the very definition of negativity means that, in terms of the eigenvalues $\{\lambda_i\}$ of the partial transpose, it can be written as

$$\mathcal{N}_A(\rho) = \sum_i \frac{|\lambda_i| - \lambda_i}{2} \tag{3.63}$$

So in that case computing the negativity is tantamount to finding the eigenvalues of ρ, which, although may not be easy as well, is certainly much better than the optimization problems involved in computing the relative entropy of entanglement or the entanglement of formation.

3.4 Correlations and Information

We now turn to the idea of correlations and how to quantify information. Our starting point will be the von-Neumman entropy, which we know to give the uncertainty on the specification of the state of a system.

First we ask what is correlation? The idea of correlation is very simple: two subsystems of a system are correlated when knowledge about one of them implies some amount of knowledge about the other. One easy example lies in conservation laws - if one whole system has conserved additive energy E_0 and one measures the energy of one subsystem and finds value E_A, then we knows, without the need of conducting a separate experiment, that the other will have energy $E_0 - E_A$. This implies in a constraint on the second system. If the systems were classical, this means that the second system would have its state constrained to be on a specific energy surface on phase space, thereby reducing the uncertainty about it. In the quantum situation, this would mean that the state must be one eigenstate of the Hamiltonian, thereby reducing to an eigenspace.

Now we shall discuss one way to quantify the *total correlation* on a state given a specific bipartition. The idea is as follows. Suppose ρ is a state on $\mathscr{H}$ and that this

Hilbert space decomposes in some physically meaningful way as $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$.

Observers restricted to either the A or B part will describe what they observe with the reduced states $\rho_A = \operatorname{Tr}_B \rho$ and $\rho_B = \operatorname{Tr}_A \rho$. Now the central idea is: recall that $S(\rho)$ is the *uncertainty in* ρ. Intuitively speaking, when we form the reduced states we are throwing information away. In that sense, interpreting $S(\rho_A) + S(\rho_B)$ as the *total uncertainty when one knows the partial states separately.* We intuitively expect that $S(\rho_A) + S(\rho_B)$ is bigger than or equal to $S(\rho)$ - since information has been thrown away in the process the uncertainty can only rise. Furthermore, the difference should be seen as the information lost, which should be a measure of the total correlations among the two parts. This motivates the definition bellow

Definition 3.4.1. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ be the Hilbert space of a quantum system. Let ρ_{AB} be a quantum state and let $\rho_A = \operatorname{Tr}_A \rho_{AB}$ and $\rho_B = \operatorname{Tr}_A \rho_{AB}$ be the partial states. We define the **mutual information** to be

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{3.64}$$

The expectation that $I(\rho_{AB}) \geq 0$ is indeed realized with this definition. Indeed the inequality

$$S(\rho_A) + S(\rho_B) \geq S(\rho_{AB}), \tag{3.65}$$

is known as the weak subadditivity of the entanglement entropy and its proof can be found in [41].

Another way to phrase exactly the same idea we have presented above is the following: if the real state of the system is ρ_{AB}, how much information we loose by believing the state to be $\rho_A \otimes \rho_B$, i.e, by just knowing the reduced states? The answer is given by the relative entropy $S(\rho_{AB}|\rho_A \otimes \rho_B)$ which can be seen to be equal to $I(\rho_{AB})$.

The mutual information quantifies *total correlations*. For this measure of correlation it does not matter whether the origin of the correlation is classical or quantum, via entanglement for example. It is possible to split $I(\rho_{AB})$ into a part which we can call classical, and a part which we can call quantum. In particular, entanglement is also captured inside the quantum part. To motivate the definition of the classical part we can ask the following question "how much can we learn about the state of A if we measure one observable which is local to B"? To answer this question, let $\{\mathbf{1} \otimes \Pi_\lambda : \lambda \in \Lambda\}$ be a projective measurement local to B. Prior to measurement the state of A is the reduced density matrix $\rho_A = \operatorname{Tr}_B \rho_{AB}$. Now the measurement is carried out. The probability of result $\lambda \in \Lambda$ is of course

$$p(\lambda) = \operatorname{Tr}(\mathbf{1} \otimes \Pi_\lambda)\rho_{AB}. \tag{3.66}$$

Furthermore if the obtained result is λ the system collapses to the state

$$\rho^\lambda_{AB} = \frac{(\mathbf{1} \otimes \Pi_\lambda)\rho_{AB}}{p(\lambda)}. \tag{3.67}$$

Let us now create a classical random variable X with the following idea: its values are parameterized by the elements of Λ and for each $\lambda \in \Lambda$ its value is the information gained about A if the measurement on B gave result λ. This can be quantified as follows: prior to measurement the state of A is ρ_A and its uncertainty is $S(\rho_A)$. After the measurement, if the result was λ the state of A is $\rho^\lambda_A = \operatorname{Tr}_B \rho^\lambda_{AB}$ and its uncertainty is $S(\rho^\lambda_A)$. The information gained is of course the difference between these uncertainties:

$$X_\lambda = S(\rho_A) - S(\rho^\lambda_A). \tag{3.68}$$

The probability of X attaining the value X_λ is of course $p(\lambda)$. Hence the mean value of X, which is nothing more than *the mean information about A gained by performing the measurement* Π *on B is*:

$$J_\Pi(A,B) = S(\rho_A) - \sum_{\lambda\in\Lambda} p(\lambda) S(\rho_A^\lambda). \tag{3.69}$$

This depends on the measurement but not on the result of the measurement. It is the mean decrease in uncertainty of A that the measure is capable of affecting after it has taken place. Finally, noticing that the above quantity depends on a specific measurement on B, we might ask whether there is another measurement that when carried out yields more information about A. Because of that, in order to capture the total locally available information, we maximize over all possible measurements on B. The resulting object is the maximum mean information we are able to get about A by measuring B. Due to this reason it is common to call this quantity as the *locally accessible information*.

Here we are going to consider just an example to illustrate the point. Consider a system of two spin-half particles and isolate their spin degrees of freedom. We have two systems of two levels. Let spin down be denoted by $|0\rangle$ and spin up be denoted by $|1\rangle$. Further denote $|a\rangle \otimes |b\rangle = |ab\rangle$. Consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle). \tag{3.70}$$

Its density matrix is of course

$$\rho = \frac{1}{2}(|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + |10\rangle\langle 10|). \tag{3.71}$$

Next consider the following state:

$$\rho_0 = \frac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 01|). \tag{3.72}$$

What is the difference between them? The state given by Eq. (3.71) is entangled, while the state given by (3.72), is separable. Still, the maximum of $J_{\Pi}(A, B)$ over all measurements Π in B *are the same*. What this example shows is even though the first state has a new purely quantum correlation, the measure we have defined only captures the classical part of the correlation.

All of this has the objective of introducing the definition bellow

Definition 3.4.2. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ be the Hilbert space of a quantum system with state ρ_{AB}. The classical correlations of ρ_{AB} by measurements in B is defined to be

$$J^{\leftarrow}(\rho_{AB}) = \max_{\Pi} \left[S(\rho_A) - \sum_{\lambda \in \Lambda} p(\lambda) S(\rho_A^{\lambda}) \right], \tag{3.73}$$

where the maximum is taken over all projective measurements local to B.

Definition 3.4.3. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ be the Hilbert space of a quantum system with state ρ_{AB}. The *quantum discord* of ρ_{AB} by measurements in B is defined to be

$$\mathcal{D}^{\leftarrow}(\rho_{AB}) = I(\rho_{AB}) - J^{\leftarrow}(\rho_{AB}), \tag{3.74}$$

where $I(\rho_{AB})$ is the mutual information of ρ_{AB} and $J^{\leftarrow}(\rho_{AB})$ is the classical correlation of ρ_{AB} by measurements in B.

We notice that in the notation, the arrow indicates which subsystem is being measured by pointing away from it in order the bipartition is made. So if in a state ρ_{AB} one measures A instead, we would denote the correlatons by $J^{\rightarrow}(\rho_{AB})$ and $\mathcal{D}^{\rightarrow}(\rho_{AB})$.

4

Quantum Field Theory and the Unruh Effect

In this chapter we review quantum field theory in curved, globally hyperbolic spacetimes and Unruh's derivation of the thermal nature of the Minkowski vacuum when probed by one Rindler observer. Since we are going to work in globally hyperbolic spacetimes we shall always assume the existence of a Cauchy surface for spacetime. We refer the reader to Appendix B for some details and definitions regarding globally hyperbolic spacetimes.

4.1 Classical Field Theory

4.1.1 Brief Review of Classical Mechanics

We first review the classical field theory of a scalar field in a globally hyperbolic spacetime. We are here focusing on globally hyperbolic spacetimes in order to have a well-defined initial value problem. Recall that a classical field theory, like electromagnetism, can be seen as a classical theory with infinitely many degrees of freedom, one for each point in space. Due to that we review some basic aspects of the setup of classical mechanics with finitely many degrees of freedom.

Suppose we have some classical system described by a finite dimensional configuration manifold Q. The configurations can be ascribed coordinates $q^1, \ldots, q^n$. The dynamics of the system is captured by specifying a path in Q, which in turn can be described by its coordinates $q^1(t), \ldots, q^n(t)$. There are two approaches to the basic dynamical problem. The Lagrangian and the Hamiltonian. In the Lagrangian approach one works in the tangent bundle TQ which has natural coordinates $q^1, \ldots, q^n$ and $\dot{q}^1, \ldots, \dot{q}^n$. The Lagrangian is then a function $L : \mathbb{R} \times TQ \to \mathbb{R}$ such that we can define the action functional S which acts on paths. It is

$$S[\gamma] = \int_a^b L(t, \gamma(t), \gamma'(t))dt. \tag{4.1}$$

The equations of motion then follow from the *principle of stationary action*. In other words, the time evolution is given as the path satisfying $\delta S = 0$. This yields the Euler-Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}. \tag{4.2}$$

The second approach is the Hamiltonian approach. Here, one works in the cotangent bundle T^*Q which has natural coordinates $q^1, \ldots, q^n$ and $p_1, \ldots, p_n$. The observables of the theory are functions defined on T^*Q which are in turn functions of the coordinates and momenta. One of the most important observables, in a sense, is the Hamiltonian, defined by a Legendre transform of the Lagrangian,

$$H = \sum p_i \dot{q}^i - L. \tag{4.3}$$

The space $P = T^*Q$ is called the phase space of the theory and carries a natural structure of a *sympletic space*. It is an even dimensional manifold, and we can define a special 2-form on it by

$$\Omega = d\left(\sum p_i dq^i\right) = \sum dp_i \wedge dq^i. \tag{4.4}$$

This form is closed and non-degenerate, *i.e.*, $d\Omega = 0$ and if $\Omega(v, w) = 0$ for all v, then $w = 0$. The last condition implies that any one-form α can be computed with Ω. To understand this, if ω is a k-form and v a vector, define the interior product $\iota_v \omega$ to be the $(k-1)$-form given by

$$\iota_v \omega(w_1, \ldots, w_{k-1}) = \omega(v, w_1, \ldots, w_{k-1}). \tag{4.5}$$

Now consider the mapping $v \mapsto \iota_v \Omega$. For any v this gives a one-form. Furthermore, suppose that $\iota_v \Omega = 0$. This means that $\Omega(v, w) = 0$ for any w. Since Ω is non-degenerate, $w = 0$. This is sufficient to establish that the mapping is an isomorphism. Spoken more simply, for any one-form α, there is a unique v such that $\alpha = \iota_v \Omega$. Now let $f \in C^\infty(P)$ be a smooth function in the phase-space. The exterior derivative df is a one-form. Hence there is a unique vector field associated to it, call it X_f, such that $df = \Omega(X_f, \cdot)$. With this structure one finally defines the *Poisson bracket* to be the mapping taking observables to

$$\{f, g\} = \Omega(X_f, X_g). \tag{4.6}$$

It is then easy to see that the Poisson bracket between any pair of coordinates and conjugate momenta reduces to a simple form

$$\{q^i, p_j\} = \delta^i_j, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0. \tag{4.7}$$

4.1.2 Classical Fields in Globally Hyperbolic Spacetimes

With this brief review of the formalisms of classical mechanics, we turn to classical field theory. To understand the theory as simply a classical theory with infinitely many degrees of freedom, fix a Cauchy surface $\Sigma \subset M$ so that we have a foliation $M \simeq \mathbb{R} \times \Sigma$. In that case, we can pick a time function, so that any point can be mapped to a pair $(t, \mathbf{x})$ with $\mathbf{x} \in \Sigma$. The classical field thus becomes $\phi(t, \mathbf{x})$. We can regard $\mathbf{x}$ as the label, as the discrete i in the classical mechanics example. So we have infinitely many coordinates evolving with time, where the time direction is given by the time vector of the foliation. We can further add a discrete label to the fields, on which case we denote the fields collectively by ϕ^a. Notice that in this case, comparing to classical mechanics, we have still infinitely many coordinates but now identified by two labels: a discrete and a continuous one.

A classical field theory is still specified by one action $S[\phi^a]$, which is written in terms of a local Lagrangian density which we assume to be constructed out of ϕ^a and the first-order derivatives

$$S[\phi^a, \nabla_\mu \phi^a] = \int_M \mathcal{L}[\phi^a, \nabla_\mu \phi^a] \epsilon \tag{4.8}$$

where the integration is over the whole spacetime M and ϵ is its volume form. The equations of motion for this theory are the Euler-Lagrange equations obtained by setting $\delta S = 0$. The obtained equations are

$$\frac{\partial \mathcal{L}}{\partial \phi^a} = \nabla_\mu \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi^a)}. \tag{4.9}$$

Even though up to this point we made a fairly general description, we shall mostly focus on the real scalar field, which is specified by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\nabla_\mu \phi \nabla^\mu \phi - (m^2 + \xi R)\phi^2), \tag{4.10}$$

where R is the Ricci scalar of the spacetime. Recall that ∇_μ is the covariant derivative operator with respect to the coordinate vector field ∂_μ and $\nabla^\mu = g^{\mu\nu}\nabla_\nu$ where g is the spacetime metric. The constant ξ is a coupling between the field and the spacetime curvature. When $\xi = 0$ we say that the field is minimally coupled, and that is the case which will interest us the most for its simplicity. The equations of motion for such a scalar field are the well-known Klein-Gordon equations,

$$(\Box - m^2)\phi = 0. \tag{4.11}$$

Now suppose the Cauchy surface Σ has normal n^μ. These equations admit one well-posed initial value formulation provided one specifies in Σ, two initial data, $f, g \in C_0^\infty(\Sigma)$, which are smooth and of compact support, such that the solution ϕ satisfies [39]:

$$\phi|_\Sigma = f, \quad n^\mu \nabla_\mu \phi|_\Sigma = g. \tag{4.12}$$

Finally, the (complexified) space of solutions to the Klein-Gordon equation carries a bilinear form, which is almost an inner product. It is defined by

$$(\phi, \psi) = i \int_{\Sigma} (\phi^* \nabla_\mu \psi - \psi \nabla_\mu \phi^*) n^\mu d\Sigma. \tag{4.13}$$

where the integration is taken over a Cauchy surface Σ. We call attention that in our conventions, the inner product is antilinear in the *first entry* as done in [39] instead of in the second, as some authors choose [5, 22]. This convention affects some signs and some complex conjugations.

It can be shown using Stoke's theorem and the Klein-Gordon equation that this bilinear form is independent of the Cauchy surface [39, 7]. Nevertheless, we say it is almost an inner product, because although it satisfies all properties required, it is not positive definite. There are subspaces of the space of solutions on which the form becomes positive-definite however, and gives rise to an inner product [38].

A special situation on which this happens is when one considers a subspace of positive frequency solutions with respect to a timelike vector field. To make this precise, a solution to the Klein-Gordon equation is called *monochromatic and positive frequency*, with respect to a timelike future-directed vector field Z, when there's $\omega \in [0, +\infty]$ with

$$\mathfrak{L}_Z \phi = -i\omega\phi, \tag{4.14}$$

where $\mathfrak{L}_Z$ denotes the Lie derivative with respect to the vector field Z, which when acting upon scalars is just the usual action of the derivation Z upon the scalar field [33].

In a subspace spanned by such kind of solutions the form is indeed positive-definite and gives rise to a true inner product.

With these remarks we finish this brief overview of the classical aspects of the Klein-Gordon field in a globally hyperbolic spacetime. Next we quickly review how the quantization is carried out in Minkowski Spacetime.

4.2 Free Quantum Field Theory in Minkowski Spacetime

Here we shall make a brief review of the free quantum field theory in Minkowski spacetime. There are many approaches to this subject. Perhaps, the most appealing, from a logical point of view, is that of Weinberg and Duncan [40, 12] in which one starts with a relativistic theory of particles and finds out that the introduction of fields is a way to construct a Lorentz invariant scattering theory satisfying the clustering requirement, i.e., that spatially separated experiments yield uncorrelated results. There are two reasons why we choose a distinct approach here. The first of them is because we are mainly interested in quantum fields in curved spacetimes. As we shall see, in that scenario, the idea of particles looses its centrality. So a *field first point of view* is conceptually helpful given our objectives. The second reason is for simplicity, since this approach is more straightforward.

Recall that in the canonical quantization of a classical system, one constructs a quantum theory following the postulates of 3, with the property that for every classical observable f, there should correspond a quantum observable $\hat{f}$ with property that

$$[\hat{f}, \hat{g}] = i\{f, g\}. \tag{4.15}$$

In particular if the system has coordinates q^i and canonically conjugate momenta p_j, it means that we should have

$$[\hat{q}^i, \hat{p}_i] = i\delta^i_j, \tag{4.16}$$

which are the famous canonical commutation relations.

We now turn to the field situation, with the classical theory structure laid out in the previous section. The canonical quantization of such a theory therefore means that we look for a quantum theory in the sense of Chapter 3 containing observables $\phi_a(x), \pi_b(x)$ with $x \in \Sigma$ where Σ is a Cauchy surface, obeying the relations

$$[\phi_a(x), \pi_b(y)] = i\delta_{ab}\delta(x-y), \quad [\phi_a(x), \phi_b(y)] = [\pi_a(x), \pi_b(y)] = 0, \quad \forall x, y \in \Sigma. \tag{4.17}$$

It is important to emphasize that these are *Schrödinger picture operators*. Up to this point they are time-independent, defined on a Cauchy surface, and the time evolution is to be attributed to the states. The Heisenberg evolution of these operators, however, is exactly given by the field equations of the classical theory.

Now, we shall, as done in [26, 32, 5], consider how this quantization is effectively carried out for the scalar field. The usual procedure is to look for solutions to the classical field equations which are positive frequency with respect to the inertial reference frame ∂_t. This means we should find solutions satisfying

$$\frac{\partial \phi}{\partial t} = -i\omega\phi. \tag{4.18}$$

These solutions will be plane waves, $\phi_{\mathbf{k}}(t, \mathbf{x}) = A_{\mathbf{k}} e^{-i\omega t} e^{i\mathbf{k}\cdot\mathbf{x}}$, with the condition that $\omega^2 = |\mathbf{k}|^2 + m^2$. We further normalize these solutions with respect to the Klein-Gordon inner product. Choosing $t = 0$ as the Cauchy surface this gives

$$\begin{aligned}(\phi_{\mathbf{k}}, \phi_{\mathbf{k}'}) &= i\int_\Sigma (\phi_{\mathbf{k}}^* \nabla_\mu \phi_{\mathbf{k}'} - \phi_{\mathbf{k}'} \nabla_\mu \phi_{\mathbf{k}}^*) n^\mu d\Sigma \\ &= i\int_{\mathbb{R}^3} d^3\mathbf{x} (\phi_{\mathbf{k}}^* \partial_t \phi_{\mathbf{k}'} - \phi_{\mathbf{k}'} \partial_t \phi_{\mathbf{k}}^*) \\ &= i\int_{\mathbb{R}^3} d^3\mathbf{x} (-i\omega_{\mathbf{k}'} \phi_{\mathbf{k}}^* \phi_{\mathbf{k}'} - i\omega_{\mathbf{k}} \phi_{\mathbf{k}}^* \phi_{\mathbf{k}'})\end{aligned} \tag{4.19}$$

Since everything is evaluated at $t = 0$ Eq. (4.19) gives

$$
\begin{aligned}
(\phi_{\mathbf{k}}, \phi_{\mathbf{k}'}) &= i \int_{\mathbb{R}^3} d^3\mathbf{x}(-i\omega_{\mathbf{k}'} A^*_{\mathbf{k}} A_{\mathbf{k}'} e^{(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} - i\omega_{\mathbf{k}} A^*_{\mathbf{k}} A_{\mathbf{k}'} e^{(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}) \\
&= A^*_{\mathbf{k}} A_{\mathbf{k}'} \int_{\mathbb{R}^3} d^3\mathbf{x}(\omega_{\mathbf{k}'} + \omega_{\mathbf{k}}) e^{(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}} \\
&= A^*_{\mathbf{k}} A_{\mathbf{k}'} (2\pi)^3 (\omega_{'} + \omega_{\mathbf{k}}) \delta(\mathbf{k}' - \mathbf{k}).
\end{aligned} \tag{4.20}
$$

Finally, the Dirac delta $\delta(\mathbf{k}' - \mathbf{k})$ forces that $\mathbf{k} = \mathbf{k}'$ and consequently one gets the result that

$$
(\phi_{\mathbf{k}}, \phi_{\mathbf{k}'}) = |A_{\mathbf{k}}|^2 (2\pi)^3 2\omega_{\mathbf{k}} \delta(\mathbf{k}' - \mathbf{k}). \tag{4.21}
$$

Hence if we want an orthonormal set with continuum normalization

$$
(\phi_{\mathbf{k}}, \phi_{\mathbf{k}'}) = \delta(\mathbf{k}' - \mathbf{k}), \tag{4.22}
$$

then it is clear from the above developments we need to take

$$
A_{\mathbf{k}} = \frac{1}{\sqrt{(2\pi)^3 (2\omega_{\mathbf{k}})}}. \tag{4.23}
$$

In doing so we conclude that we have an orthonormal set of solutions to the Klein-Gordon equation, which are positive frequency with respect to the inertial reference frame time. This set of solutions is

$$
\phi_{\mathbf{k}}(x) = \frac{1}{\sqrt{(2\pi)^3 (2\omega_{\mathbf{k}})}} e^{ikx}, \tag{4.24}
$$

where, in the exponent, $kx = k_\mu x^\mu$ and we have defined the four-vector $k = (\omega_{\mathbf{k}}, \mathbf{k})$.

The next step is to expand a general solution to the Klein-Gordon equation as a combination of such elementary solutions, together with their negative frequency counterparts $\phi^*_{\mathbf{k}}$:

$$\phi(x) = \int d^3\mathbf{k}[a(\mathbf{k})\phi_{\mathbf{k}}(x) + a^*(\mathbf{k})\phi^*_{\mathbf{k}}(x)]. \tag{4.25}$$

It is furthermore clear that due to the orthonormality of the solutions, that we can extract a and a^* from the field as

$$a(\mathbf{k}) = (\phi_{\mathbf{k}}, \phi), \quad a^*(\mathbf{k}) = -(\phi^*_{\mathbf{k}}, \phi). \tag{4.26}$$

Now, if one wants to realize ϕ, ϕ^*, and the corresponding conjugate momenta, as operators on a Hilbert space satisfying the CCR, while keeping ϕ being a solution to the Klein-Gordon equation, then it is clear that what should be promoted to operators are the $a(\mathbf{k}), a^*(\mathbf{k})$ functions. Upon transforming these into operators the complex conjugation turns to the adjoint operation and we get the pair $a(\mathbf{k}), a^\dagger(\mathbf{k})$. Furthermore, a standard computation shows that the CCR will hold, if and only if, the relations between $a(\mathbf{k}), a^\dagger(\mathbf{k})$ are:

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = i(\phi_{\mathbf{k}}, \phi_{\mathbf{k}'}), \quad [a(\mathbf{k}), a(\mathbf{k}')] = [a^\dagger(\mathbf{k}), a^\dagger(\mathbf{k}')] = 0. \tag{4.27}$$

These are commutation relations between creation and annihilation operators on a Fock space. They give rise to the canonical quantization of a field, namely, the Fock space picture. In fact, the idea is that for each $\mathbf{k}$ we have a harmonic oscilator with ladder operators $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$. We then simply have a state, called the vacuum and denoted by $|0\rangle$, with the property that

$$a(\mathbf{k})|0\rangle = 0, \tag{4.28}$$

and the Hilbert space is a Fock space built by the operators $a^\dagger(\mathbf{k})$ acting on this vacuum.

Now, the reason why the resulting construction can be interpreted in terms of particles comes from evaluating the energy-momentum tensor and so we give a quick review of how this is usually done.

4.2.1 Observables and the Energy-Momentum Tensor

In the previous subsection we reviewed that a quantum field may be quantized by expanding it into modes of positive frequency with the expansion coefficients behaving as creation and annihilation operators. Nevertheless, one needs to interpret the construction outlined, and of course that must be done by studying the observables of the theory. In the case of a field, the central observable is the energy-momentum tensor.

The energy-momentum tensor for the Klein-Gordon field can be obtained from the general definition given in Eq. (2.8), which we rewrite below,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{4.29}$$

In that equation S_M is the matter action, which in the case of the Klein-Gordon field, is just the Klein-Gordon action. For the real, minimally-coupled Klein-Gordon action, the above definition may be seen to give [39, 7]:

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\beta\phi + m^2\phi^2 \tag{4.30}$$

In particular, since we are working in the Minkowski spacetime and we have the inertial reference frame at our disposal, we can decompose the energy-momentum tensor unambiguously into the four-momentum and four-angular momentum. We shall focus on the four-momentum here. The four-momentum with respect to the inertial reference frame is:

$$P^\mu = \int_\Sigma T^{\mu 0} d\Sigma = \int_{\mathbb{R}^3} T^{\mu 0}(0, \mathbf{x}) d^3\mathbf{x}. \tag{4.31}$$

where Σ is the $t = 0$ Cauchy surface.

A standard and straightforward but lengthy computation, which we shall not carry out here, allows us to explicitly compute P^μ in terms of the mode expansion of the Klein-Gordon field, where the mode functions are the previous ones, positive frequency with respect to the inertial observer. The result is that we have [5, 26, 32]:

$$P^0 = \int \omega_{\mathbf{k}} a^\dagger(\mathbf{k}) a(\mathbf{k}) d^3\mathbf{k}, \quad P^i = \int \mathbf{k} a^\dagger(\mathbf{k}) a(\mathbf{k}) d^3\mathbf{k}. \tag{4.32}$$

This result allows the interpretation that $a^\dagger(\mathbf{k})$ adds a particle with momentum $\mathbf{k}$ and energy $\omega_{\mathbf{k}}^2 = m^2 + |\mathbf{k}|^2$. In that sense, the decomposition into positive frequency modes with respect to the inertial Minkowski observer allows for a particle interpretation with respect to himself.

4.3 Quantum Field Theory in Curved Spacetimes

With this brief review of Quantum Field Theory in Minkowski Spacetime, we turn to the curved spacetime case. This is a semiclassical approximation on which we treat all matter with the laws of quantum mechanics and yet treat gravity classically in the framework of General Relativity. It becomes important when describing quantum phenomena on situations on which the gravitational interaction is still important. The main example, which is of interest to us, is in the vicinity of a black hole.

While Classical Field Theory has a straightforward extension to curved spacetimes - actually so straigthforward that it is natural to directly formulate the theory in the most general setting - Quantum Field Theory has not. Indeed, as just reviewed, the textbook approach to Quantum Field Theory in Minkowski spacetime relies in a decomposition of the field into modes of positive and negative frequencies, as seen by the inertal observer, accompanied by the annihilation and creation operators. This procedure works in the Minkowski spacetime because there is a preferred time, that of the inertial observer, which in turn is associated to the Poincaré symmetry of spacetime. This singles out a way to split a solution to the classical field equations into positive and negative frequencies. In a general spacetime that will not be the case and there will not exist a preferred choice.

As many authors have pointed out [5, 38, 25], it is not that there is no way of choosing a decomposition into positive and negative frequencies. Instead there are many ways and none seems to be preferred in general. The reason why this is problematic is that the theories constructed using two different such choices in general are unitarilly inequivalent, so that one gets distinct quantum theories which cannot be mapped into each other by some unitary transformation [38].

A powerful and modern approach to the subject is the so-called algebraic quantum field theory, reviewed for example in [18]. This approach is particularly well-suited for investigating quantum field theory in curved spacetimes because it allows one to systematically tackle these different constructions of the quantum theory, without committing to one beforehand. Even though it has a lot of benefits, we shall not use it here. Instead we shall follow the standard old-fashioned approach just for simplicity. This approach is a straightforward generalization of what we do in Minkowski spacetime. We shall, again for simplicity, consider a scalar field.

To quantize the field, we again shall express it in terms of a complete orthonormal set of mode solutions $\{u_i, u_i^*\}$, where the $\{u_i\}$ is a set of monochromatic positive frequency solutions with respect to some timelike vector field. The orthonormality of the set is expressed by the relations

$$(u_i, u_j) = \delta_{ij}, \quad (u_i^*, u_j^*) = -\delta_{ij}, \quad (u_i, u_j^*) = 0. \tag{4.33}$$

Notice that the orthonormality condition for the negative frequency modes carries a minus sign, which is considerably important.

A general solution ϕ is expanded as

$$\phi = \sum a_i u_i + b_i^\dagger u_i^*. \tag{4.34}$$

We can further find out what a_i and b_i are in terms of the inner product. Taking the inner product, (u_k, ϕ), we get

$$\begin{aligned}
(u_k, \phi) &= \left(u_k, \sum_i a_i u_i + b_i^\dagger u_i^*\right) \\
&= \sum_i a_i (u_k, u_i) + b_i^\dagger (u_k, u_i^*) \\
&= \sum_i a_i \delta_{ik} \\
&= a_k.
\end{aligned} \tag{4.35}$$

In the same way we get

$$\begin{aligned}
(u_k^*, \phi) &= \left(u_k^*, \sum_i a_i u_i + b_i^\dagger u_i^*\right) \\
&= \sum_i a_i (u_k^*, u_i) + b_i^\dagger (u_k^*, u_i^*) \\
&= \sum_i -b_i^\dagger \delta_{ik} \\
&= -b_i^\dagger.
\end{aligned} \tag{4.36}$$

In that sense we conclude that we have the expansion

$$a_i = (u_i, \phi), \tag{4.37a}$$

$$b_i^\dagger = -(u_i^*, \phi). \tag{4.37b}$$

To quantize the field, one promotes $a_i, a_i^\dagger$ and $b_i, b_i^\dagger$ to operators satisfying the commutation relations

$$[a_i, a_j^\dagger] = [b_i, b_j^\dagger] = \delta_{ij} \tag{4.38}$$

with all other commutators vanishing. But now this construction is actually far from unique. We can consider now a second set $\{\overline{u}_j, \overline{u}_j^*\}$ and again decompose the field:

$$\phi = \sum \overline{a}_j \overline{u}_j + \overline{b}_j^\dagger \overline{u}_j^*. \tag{4.39}$$

The first construction yields a vacuum $|0\rangle$ and the second construction yields a vacuum $|\overline{0}\rangle$, each of which with their associated Fock spaces.

The central aspect of this approach is to now relate the two constructions. Since the $\{u_i, u_i^*\}$ set is complete, they can be used to expand the mode functions $\overline{u}_j$ and $\overline{u}_j^*$. We obtain

$$\overline{u}_j = \sum_i \alpha_{ji} u_i + \beta_{ji} u_i^*, \tag{4.40a}$$

$$\overline{u}_j^* = \sum_i \beta_{ji}^* u_i + \alpha_{ji}^* u_i^*. \tag{4.40b}$$

Our first objective is to find out in general what are the coefficients using the Klein-Gordon inner product. We have already done that for a general solution, and using the same Eqs. (4.37a) we have

$$\alpha_{ji} = (u_i, \overline{u}_j), \quad \beta_{ji} = -(u_i^*, \overline{u}_j). \tag{4.41}$$

In the same way we can perform the expansion of u_i and u_i^* in terms of the basis $\{\overline{u}_j, \overline{u}_j^*\}$. It is

$$u_i = \sum_j \gamma_{ij}\overline{u}_j + \eta_{ij}\overline{u}_j^*, \tag{4.42a}$$

$$u_i^* = \sum_j \eta_{ij}^*\overline{u}_j + \gamma_{ij}^*\overline{u}_j^*. \tag{4.42b}$$

The expansion coefficients γ_{ij} and η_{ij} are all related to the previous ones α_{ij} and β_{ij}, defined by Eqs. (4.41), in the expansion of $\overline{u}_j$ and $\overline{u}_j^*$ by virtue of the properties of the inner product. To see this, using the same equations to extract the coefficients in the expansion of a solution of the Klein-Gordon equation, we have

$$\gamma_{ij} = (\overline{u}_j, u_i), \tag{4.43a}$$

$$\eta_{ij} = (\overline{u}_j^*, u_i). \tag{4.43b}$$

Now to relate γ_{ij} and η_{ij} to α_{ji} and β_{ji} we shall discuss a property specific to the Klein-Gordon inner product. Consider the evaluation of (f, g), and the process of exchanging the order of f and g inside the integration. It is

$$\begin{aligned}(f, g) &= i\int_\Sigma (f^*\nabla_\mu g - g\nabla_\mu f^*)n^\mu d\Sigma \\ &= -i\int_\Sigma (g\nabla_\mu f^* - f^*\nabla_\mu g)n^\mu d\Sigma \\ &= -(g^*, f^*),\end{aligned} \tag{4.44}$$

where in passing from the second line to the third we have recalled that $g = (g^*)^*$. So we can commute the functions inside the inner product, at the price of taking the complex conjugate and adding a minus sign. This allows us to obtain γ_{ij} and η_{ij} in terms of α_{ji} and β_{ji}:

$$\gamma_{ij} = (\overline{u}_j, u_i) = -(u_i^*, \overline{u}_j^*) = \alpha_{ji}^*, \tag{4.45a}$$

$$\eta_{ij} = -(\overline{u}_j^*, u_i) = (u_i^*, \overline{u}_j) = -\beta_{ji}. \tag{4.45b}$$

Therefore we can dispense with the γ_{ij} and η_{ij} and write everything in terms of the coefficients α_{ji} and β_{ji}, or vice versa. To summarize, the result is that we have the transformations

$$\overline{u}_j = \sum_i \alpha_{ji} u_i + \beta_{ji} u_i^*, \quad \overline{u}_j^* = \sum_i \beta_{ji}^* u_i + \alpha_{ji}^* u_i^*, \tag{4.46a}$$

$$u_i = \sum_j \alpha_{ji}^* \overline{u}_i - \beta_{ji} \overline{u}_i^*, \quad u_i^* = \sum_j -\beta_{ji}^* \overline{u}_i + \alpha_{ji} \overline{u}_i^*. \tag{4.46b}$$

These transformations are called *Bogolubov transformations* and the coefficients α_{ji} and β_{ji} are called *Bogolubov coefficients*. They enjoy two special properties. The first follows from $(\overline{u}_i, \overline{u}_j) = \delta_{ij}$. This equation yields

$$\begin{aligned}
(\overline{u}_i, \overline{u}_j) &= \left(\sum_\ell \alpha_{i\ell} u_\ell + \beta_{i\ell} u_\ell^*, \sum_k \alpha_{jk} u_k + \beta_{jk} u_k^* \right) \\
&= \sum_{\ell,k} \alpha_{i\ell}^* \alpha_{jk} (u_\ell, u_k) + \alpha_{i\ell}^* \beta_{jk} (u_i, u_k^*) + \beta_{i\ell}^* \alpha_{jk} (u_\ell^*, u_k) + \beta_{i\ell}^* \beta_{jk} (u_\ell^*, u_k^*) \\
&= \sum_{\ell,k} \alpha_{i\ell}^* \alpha_{jk} \delta_{ik} - \beta_{i\ell}^* \beta_{jk} \delta_{\ell k} \\
&= \sum_\ell \alpha_{i\ell}^* \alpha_{j\ell} - \beta_{i\ell}^* \beta_{j\ell} \\
&= \delta_{ij}.
\end{aligned} \tag{4.47}$$

A second property follows from $(\overline{u}_i, \overline{u}_j^*) = 0$. Writing it in terms of Bogolubov coefficients it yields

$$\begin{aligned}
(\overline{u}_i, \overline{u}_j^*) &= \left(\sum_\ell \alpha_{i\ell} u_\ell + \beta_{i\ell} u_\ell^*, \sum_k \beta_{jk}^* u_k + \alpha_{jk}^* u_k^* \right) \\
&= \sum_{\ell,k} \alpha_{i\ell}^* \beta_{jk}^* (u_\ell, u_k) + \alpha_{i\ell}^* \alpha_{jk}^* (u_\ell, u_k^*) + \beta_{i\ell}^* \beta_{jk}^* (u_\ell^*, u_k) + \beta_{i\ell}^* \alpha_{jk}^* (u_\ell^*, u_k^*) \\
&= \sum_{\ell,k} \alpha_{i\ell}^* \beta_{jk}^* \delta_{\ell k} - \beta_{i\ell}^* \alpha_{jk}^* \delta_{\ell k} \\
&= \sum_\ell \alpha_{i\ell}^* \beta_{j\ell}^* - \beta_{i\ell}^* \alpha_{j\ell}^* \\
&= 0.
\end{aligned} \tag{4.48}$$

Using the Bogolubov coefficients we can easily relate the expansion of ϕ in terms of the two bases. Indeed, we have

$$\begin{aligned}
\overline{a}_j &= (\overline{u}_j, \phi) \\
&= \left(\sum_i \alpha_{ji} u_i + \beta_{ji} u_i^*, \phi \right) \\
&= \sum_i \alpha_{ji} (u_i, \phi) + \beta_{ji} (u_i^*, \phi) \\
&= \sum_i \alpha_{ji} a_i - \beta_{ji} b_i^\dagger.
\end{aligned} \tag{4.49}$$

In the same way we have

$$\begin{aligned}
\bar{b}_j^\dagger &= -(\bar{u}_j^*, \phi) \\
&= -\left(\sum_i \beta_{ji}^* u_i + \alpha_{ji}^* u_i^*, \phi\right) \\
&= -\sum_i \beta_{ji}^* (u_i, \phi) + \alpha_{ji}^* (u_i^*, \phi) \\
&= \sum_i -\beta_{ji}^* a_i + \alpha_{ji}^* b_i^\dagger. \qquad (4.50)
\end{aligned}$$

The adjoints $\bar{a}_j^\dagger$ and $\bar{b}_j$ follow easily from Eqs. (4.49) and (4.50) by taking the adjoint of the equations we have derived. We can obviously obtain $a_i, a_i^\dagger$ and $b_i, b_i^\dagger$ in terms of the barred counterparts by the same procedure.

The next natural step is to relate the vacua $|0\rangle$ and $|\bar{0}\rangle$. The strategy is to write down the defining equation for $|0\rangle$ and use the Bogolubov transformations to relate it to the Fock space construction based on $|\bar{0}\rangle$. Indeed, the defining equation is $a_i|0\rangle = 0$. Using the transformation equation we obtain

$$\left(\sum_j \alpha_{ij} a_j - \beta_{ij} b_j^\dagger\right) |\bar{0}\rangle = 0. \qquad (4.51)$$

Now the Bogolubov coefficients properties derived in Eqs. (4.47) and (4.48) imply that α_{ij} and β_{ij} have inverses [5]. In that sense, we can multiply the above equation on the left by α_{ki}^{-1} and sum over i to get

$$\left(\sum_{i,j} \alpha_{ki}^{-1} \alpha_{ij} a_j - \alpha_{ki}^{-1} \beta_{ij} b_j^\dagger\right) |\bar{0}\rangle = 0. \qquad (4.52)$$

This can be simplified further to yield

$$\left(a_k - \sum_j V_{jk} b_j^\dagger\right) |\bar{0}\rangle = 0, \quad V_{jk} = \sum_i \alpha_{ki}^{-1} \beta_{ij}. \tag{4.53}$$

Eq. (4.53) can be solved [11, 22] by means of decomposing $|\bar{0}\rangle$ into the Fock basis of $|0\rangle$ and comparing the coefficients in the expansion. The result is that one has

$$|\bar{0}\rangle = \langle \bar{0}|0\rangle \exp\left[\frac{1}{2}\sum_{ij} V_{ij} a_i^\dagger b_j^\dagger\right] |0\rangle. \tag{4.54}$$

For an outline of the proof we refer the reader to [22] and for the full details of the proof we refer the reader to [11].

4.4 The Unruh Effect

With the basic idea already organized, we set out to derive in details the so-called Unruh effect. We shall follow closely the basic approach to this derivation [5, 7, 25] instead of pursuing mathematical rigor. Our notation agrees with that of [7]. Also, for simplicity, we are going to work in $1+1$ dimensions, so that we have only one spatial dimension.

What we are going to do is to consider the right and left Rindler wedges introduced in Section 2.4 as spacetimes in their own right, with the metric inherited from the Minkowski spacetime, perform the quantization of a neutral free scalar field following the methods of the previous sections and finally compare to the quantization from the Minkowski spacetime perspective.

In the right Rindler wedge we have coordinates (η, ξ) related to Minkowski coordinates (t, x) by

$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta), \quad x = \frac{1}{a} e^{a\xi} \cosh(a\eta), \tag{4.55}$$

while in the left Rindler wedge we have coordinates (η, ξ) related to Minkowski coordinates (t, x) by

$$t = -\frac{1}{a}e^{a\xi}\sinh(a\eta), \quad x = -\frac{1}{a}e^{a\xi}\cosh(a\eta), \tag{4.56}$$

with the metric acquiring in both wedges the form

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2). \tag{4.57}$$

From now on we shall call the right and left Rindler wedges respectively as regions I and II of Rindler spacetime, or simply regions I and II.

It is important to recall that Minkowski's spacetime time-orientation is given with respect to the inertial reference frame ∂_t. So a timelike vector Z is future-directed when $g(\partial_t, Z) < 0$.

It is straightforward to check that in region I, ∂_η is indeed future-directed. That follows because

$$\frac{\partial}{\partial\eta} = \frac{\partial t}{\partial\eta}\frac{\partial}{\partial t} + \frac{\partial x}{\partial\eta}\frac{\partial}{\partial x} \tag{4.58}$$

$$= e^{a\xi}\cosh(a\eta)\frac{\partial}{\partial t} + e^{a\xi}\sinh(a\eta)\frac{\partial}{\partial x}. \tag{4.59}$$

In that case, since $g(\partial_t, \partial_t) = -1$ it follows that $g(\partial_t, \partial_\eta) = -e^{a\xi}\cosh(a\eta)$. This is always negative, and hence ∂_η is indeed future-directed. Now if the same computation is performed on region II it is straightforward to see that $g(\partial_\eta, \partial_\eta) = e^{a\xi}\cosh(a\eta)$. This is now always positive and ∂_η is past-directed. To obtain a future-directed timelike Killing field on region II we thus need to consider $-\partial_\eta$ and this will be important for quantization.

The Klein-Gordon equation for a massless scalar field becomes, in these coordinates, in either region:

$$\Box\phi = e^{-2a\xi}\left(-\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2\phi}{\partial\xi^2}\right) = 0. \tag{4.60}$$

We now will focus on region I. It is clear that the positive-frequency solutions with respect to η are parametrized by a real-number k and given by

$$g_k(\eta, \xi) = C_k e^{-i\omega\eta + ik\xi}, \quad \omega = |k|. \tag{4.61}$$

To compute the normalization C_k we must impose that the Klein-Gordon inner product satisfy

$$(g_k, g_{k'}) = \delta(k - k'). \tag{4.62}$$

To do so, we recall that the product does not depend on the Cauchy surface. So we pick as Cauchy surface exactly the surface $\eta = 0$. The normal vector is the vector physically equivalent to the exterior derivative $d\eta$. It is easily seen to be

$$n = e^{-a\xi}\partial_\eta. \tag{4.63}$$

Finally we recall that, when integrating over $\eta = 0$, the induced volume form carries the determinant of the induced metric. Hence, we might write the measure as

$$d\Sigma = e^{a\xi} d\xi. \tag{4.64}$$

Combining Eqs. (4.61), (4.63) and (4.64) and evaluating the Klein-Gordon inner product yields

$$(g_k, g_{k'}) = 4\pi |C_k|^2 \omega \delta(k' - k). \tag{4.65}$$

This immediately implies that to satisfy Eq. (4.62) we should pick the modes

$$g_k^I(\eta, \xi) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi}. \tag{4.66}$$

For region II, we need to remind ourselves that positive-frequency solutions ought to be judged with respect to $-\partial_\eta$, which is the future-directed timelike Killing vector. In that case, the exact same computation yields the normalized modes

$$g_k^{II}(\eta, \xi) = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega\eta + ik\xi}. \tag{4.67}$$

We thus further extend the modes g_k^I to region II by taking them to be zero there, and we also extend the modes g_k^{II} to region I by taking them to be zero there. With this, we can decompose the quantum field as

$$\phi = \int_{-\infty}^{\infty} b_k^{(I)} g_k^{(I)} + b_k^{(II)} g_k^{(II)} + b_k^{I\dagger} g_k^{(I)*} + b_k^{(II)\dagger} g_k^{(II)*} dk. \tag{4.68}$$

This defines creation and annihilation operators $b_k^{(\Omega)}, b_k^{(\Omega)\dagger}$ where $\Omega = I, II$. We further define the so-called Rindler vacuum by the condition

$$b_k^{(\Omega)} |0_R\rangle = 0, \quad \Omega = I, II \quad \forall k \in \mathbb{R}. \tag{4.69}$$

Our objective is to compare the vacuum $|0_R\rangle$ defined by Eq. (4.69) to the Minkowski vacuum via Bogolubov transformations. The idea is that for the inertial observer, the appropriate modes are

$$f_k(x) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega t + ikx}, \quad \omega = |k|. \tag{4.70}$$

One could proceed here by simply computing the Bogolubov coefficients by expanding the Rindler modes in terms of Minkowski modes. There is one workaround which simplifies matters, however. The method, which is due to Unruh, and presented very clearly in [7], whose approach we now follow, is to construct a new set of modes with the property that they define the same vacuum as the Minkowski modes, but which have a simpler relation to the Rindler modes. In fact, the resulting modes, called *Unruh modes* are very important in the discussion of the information theory aspects which we shall dive into in the next chapter and this importance is exactly due to the simple relation they hold with the Rindler modes.

The idea behind the construction is to start with the Rindler modes $g_k^{(I)}$ supported in the right Rindler wedge. Since this is in fact just a part of Minkowski spacetime, we can *analitically extend* these modes to the other regions of Minkowski spacetime, and in particular to the left Rindler wedge.

Since the Minkowski coordinates cover the whole Minkowski spacetime, this procedure can be carried out by simply rewriting $g_k^{(I)}$ in Minkowski coordinates and use this coordinate expression to define the extension outside of the right Rindler wedge, where the coordinates still make sense. To do so, we rewrite the coordinate transformation between Minkowski and Rindler as

$$t = \frac{1}{a}\frac{e^{a(\xi+\eta)} - e^{-a(\eta-\xi)}}{2}, \quad x = \frac{1}{a}\frac{e^{a(\xi+\eta)} + e^{-a(\eta-\xi)}}{2}, \tag{4.71}$$

where we have just explicitly written the definition of $\sinh(a\eta)$ and $\cosh(a\eta)$. Now by summing and subtracting the two expressions in Eq. (4.71) we obtain the relation

$$a(t+x) = e^{a(\xi+\eta)}, \quad a(x-t) = e^{-a(\eta-\xi)}. \tag{4.72}$$

But now notice that since we have $\omega = k$,

$$g_k^{(I)}(\eta,\xi) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta + ik\xi} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega(\eta-\xi)}. \tag{4.73}$$

We can now use Eq. (4.72) to identify

$$e^{-i\omega(\eta-\xi)} = (e^{-a(\eta-\xi)})^{i\omega/a} = a^{i\omega/a}(x-t)^{i\omega/a}. \tag{4.74}$$

The end result is that we have written the Rindler modes $g_k^{(I)}$ for $k > 0$ as

$$g_k^{(I)} = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a}(x-t)^{i\omega/a}. \tag{4.75}$$

This expression can be extended outside of the right Rindler wedge, to the whole Minkowski spacetime, as it is written in a chart covering the whole manifold.

For completeness we now turn to the same procedure applied to the $k < 0$ case which we shall deal with by still supposing $k > 0$, and considering the modes $g_{-k}^{(I)}$. In fact, for this case we have

$$g_{-k}^{(I)}(\eta,\xi) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\eta - ik\xi} = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega(\eta+\xi)}. \tag{4.76}$$

where in the last step we have used that $\omega = |k| = k$. We now Eq. (4.72) again. The appropriate transformation is

$$e^{-i\omega(\eta+\xi)} = (e^{a(\xi+\eta)})^{-i\omega/a} = a^{-i\omega/a}(t+x)^{-i\omega/a}. \tag{4.77}$$

From this we obtain the expression of the modes $g_{-k}^{(I)}$ in Minkowski coordinates

$$g_{-k}^{(I)} = \frac{1}{\sqrt{4\pi\omega}} a^{-i\omega/a}(t+x)^{-i\omega/a}, \tag{4.78}$$

which yet again can be analitically extended by using this expression for points on the Minkowski spacetime outside the right Rindler wedge.

Now the second part of the method is to understand how the resulting Eqs. (4.75) and (4.78) relate to the Rindler modes $g_k^{(II)}$ on the left Rindler wedge. To do so, we simply need to do the same procedure, and write those in terms of Minkowski coordinates. The first step is to recall that the relation between the Rindler coordinates and the Minkowski coordinates is different in that region of the manifold. In fact, explicit-ing the hyperbolic functions, it is

$$t = -\frac{e^{a(\xi+\eta)} - e^{-a(\eta-\xi)}}{2a}, \quad x = -\frac{e^{a(\xi+\eta)} + e^{-a(\eta-\xi)}}{2a}. \tag{4.79}$$

From these it follows immediately that

$$a(-t-x) = e^{a(\xi+\eta)}, \quad a(-x+t) = e^{-a(\eta-\xi)}. \tag{4.80}$$

These are sufficient to carry out the procedure for the modes supported in the left Rindler wedge. For the $k > 0$ case we have

$$g_k^{(II)} = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega\eta+ik\xi} = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega(\eta+\xi)}. \tag{4.81}$$

Using the newly derived transformations, valid for the coordinates in this region, we get

$$e^{i\omega(\eta+\xi)} = (e^{a(\eta+\xi)})^{i\omega/a} = a^{i\omega/a}(-x-t)^{i\omega/a}. \tag{4.82}$$

All this combined gives the result that the modes $g_k^{(II)}$ can be written as

$$g_k^{(II)} = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a}(-t-x)^{i\omega/a}. \tag{4.83}$$

For the $k < 0$ situation we again explicitly write $g_{-k}^{(II)}$ where $k > 0$, so that with such conventions, $\omega = |k| = k$. With these remarks We can perform the same computations

$$g_{-k}^{(II)} = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega\eta - ik\xi} = \frac{1}{\sqrt{4\pi\omega}} e^{i\omega(\eta-\xi)}. \tag{4.84}$$

Using Eq. (4.80), we obtain

$$e^{i\omega(\eta-\xi)} = (e^{-a(\eta-\xi)})^{-i\omega/a} = a^{-i\omega/a}(-x+t)^{-i\omega/a}. \tag{4.85}$$

from which it follows immediately that

$$g_{-k}^{(II)} = \frac{1}{\sqrt{4\pi\omega}} a^{-i\omega/a}(t-x)^{-i\omega/a}. \tag{4.86}$$

Now we would like to express the analytic extension of $g_k^{(I)}$ for $k > 0$ with respect to modes given by Eqs. (4.83) and (4.86). We could do so by means of the Klein-Gordon inner product, but we need not do this. In fact, for the present case this can be done by inspection. The analytic expression of $g_k^{(I)}$ in Minkowski coordinates depends on $(x-t)$. This kind of dependence is present on the modes $g_{-k}^{(II)}$ when expressed in Minkowski coordinates, which depends on $(t-x)$, we will just need to factor a minus sign. Also, since the exponents in $g_k^{(I)}$ are $+i\omega/a$ and on $g_{-k}^{(II)}$ are $-i\omega/a$ we further need a complex conjugation. In fact, taking the complex conjugate of $g_{-k}^{(II)}$ we obtain

$$g_{-k}^{(II)*} = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a}(t-x)^{i\omega/a}. \tag{4.87}$$

Further writing $(x-t) = -(t-x)$ and using $-1 = e^{-i\pi}$ we get

$$g_{-k}^{(II)*} = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a}(e^{-i\pi}(x-t))^{i\omega/a} = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} e^{\pi\omega/a}(x-t)^{i\omega/a}. \tag{4.88}$$

Therefore, it is clear that

$$e^{-\pi\omega/a} g_{-k}^{(II)*} = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} (x-t)^{i\omega/a}. \tag{4.89}$$

This means that $e^{-\pi\omega/a} g_{-k}^{(II)*}$ has the same expression in Minkowski coordinates in the left Rindler wedge as $g_k^{(I)}$ in the right Rindler wedge. Since the modes (not their analytic extensions!) $g_k^{(I)}$ are zero on the left Rindler wedge and $g_k^{(II)}$ are zero on the right Rindler wedge, it is clear that the left hand side of

$$(g_k^{(I)} + e^{-\pi\omega/a} g_{-k}^{(II)*}) = \frac{1}{\sqrt{4\pi\omega}} a^{i\omega/a} (x-t)^{i\omega/a}, \tag{4.90}$$

reduces to $g_k^{(I)}$ on the right Rindler wedge. The left hand side of this function is the expression in Rindler modes of the analytic extension of $g_k^{(I)}$. We wish to normalize it though, because it is not normalized. The normalization procedure can be carried out analogously to the cases we have already outlined, the result is that we obtain the modes

$$h_k^{(I)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\pi\omega/2a} g_k^{(I)} + e^{-\pi\omega/2a} g_{-k}^{(II)} \right). \tag{4.91}$$

The exact same procedure can be carried out for the modes $g_k^{(II)}$. In other words, we take their expression in Minkowski coordinates, find out what combination of modes $g_{k'}^{(I)}$ will reproduce the same expression in order to express the analytic expression in terms of the original Rindler modes. One finally normalizes the resulting modes. The resulting modes will be

$$h_k^{(II)} = \frac{1}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \left(e^{\pi\omega/2a} g_k^{(II)} + e^{-\pi\omega/2a} g_{-k}^{(I)} \right). \tag{4.92}$$

We have outlined the construction for the case $k > 0$ so these are the resulting modes for this condition. The other condition could then be treated analogously and then would define the $h_k^{(I)}$ and $h_k^{(II)}$ for $k < 0$.

The resulting set of modes $\{h_k^{(I)}, h_k^{(I)*}, h_k^{(II)}, h_k^{(II)*}\}$ can then be used to quantize the field. We express it as

$$\phi = \int dk(c_k^{(I)} h_k^{(I)} + c_k^{(I)\dagger} h_k^{(I)*} + c_k^{(II)} h_k^{(II)} + c_k^{(II)\dagger} h_k^{(II)*}). \tag{4.93}$$

Now, the Bogolubov coefficients relating this construction to the Rindler construction may be obtained very easily. In fact, if we write down

$$g_k^{(I)} = \int dk' (\alpha_{kk'}^{(I)} h_{k'}^{(I)} + \beta_{kk'}^{(I)} h_{k'}^{(II)}), \tag{4.94}$$

we know that we must have

$$\alpha_{kk'}^{(I)} = (h_{k'}^{(I)}, g_k^{(I)}) = \frac{e^{\pi\omega/2a}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \delta(k-k'), \tag{4.95a}$$
$$\beta_{kk'}^{(I)} = (h_{k'}^{(II)}, g_k^{(I)}) = \frac{e^{-\pi\omega/2a}}{\sqrt{2\sinh\left(\frac{\pi\omega}{a}\right)}} \delta(k-k'), \tag{4.95b}$$

so that we see the transformation is "diagonal", which makes matters much simpler.

Another important fact is that these modes are combinations of only positive frequency Minkowski modes. This is explicit from their construction and expression in Minkowski coordinates. So the vacuum of this construction is just the Minkowski vacuum.

Combining the Bogolubov coefficients given by Eqs. (4.95a) and (4.95b) with the fact that the vacuum defined by the new quantization is just the Minkowski vacuum, and finally employing Eq. (4.54) relating two vacua by Bogolubov transformations, it follows that the Minkowski vacuum can be written in the Rindler construction as

$$|0\rangle = \bigotimes_{\omega} N_{\omega} \sum_{n_{\omega}=0}^{\infty} e^{-\pi n_{\omega}\omega/a} |n_{\omega}\rangle_{\omega}^{(I)} \otimes |n_{\omega}\rangle_{\omega}^{(II)}. \tag{4.96}$$

After one traces out the modes supported in the right Rindler wedge, in order to get the effective state experienced by a Rindler observer, one gets a thermal density operator with temperature

$$T = \frac{a}{2\pi}, \tag{4.97}$$

where a is the acceleration of the Rindler observer whose worldline in Rindler coordinates is $\xi = 0$. The temperature probed a distinct Rindler observer can be computed using the following general method. Suppose we have a reference frame Z which is locally proportional to a timelike Killing vector field K, *i.e.*, there is a scalar function $V(x)$ so that

$$Z^{\mu}(x) = V(x)K^{\mu}(x). \tag{4.98}$$

Now, suppose that $\gamma : [a, b] \to M$ is a geodesic worldline of some particle with four-momentum p defined along its trajectory. Then since K is a Killing vector field and since γ is a geodesic, the quantity $E = -K_{\mu}p^{\mu}$ is constant along the trajectory.

Now one observer on the reference frame Z meets the particle at $\gamma(a)$. The energy the observer will measure is $E_a = -Z_{\mu}(\gamma(a))p^{\mu}(a)$. Next, another observer of the reference frame Z meets the particle at $\gamma(b)$. The energy this observer will measure is $E_b = -Z_{\mu}(\gamma(b))p^{\mu}(b)$. We now take E_a and write Z in terms of V and K

$$E_a = -V(\gamma(a))K_{\mu}(\gamma(a))p^{\mu}(a). \tag{4.99}$$

but since $E = -K_\mu(\gamma(\tau))p^\mu(\tau)$ is constant along the geodesic, *i.e.*, for all $\tau \in [a,b]$, we have the equality

$$-K_\mu(\gamma(a))p^\mu(a) = -K_\mu(\gamma(b))p^\mu(b). \tag{4.100}$$

Hence using Eq. (4.100) in Eq. (4.99) we obtain

$$\begin{aligned} E_a &= -V(\gamma(a))K_\mu(\gamma(b))p^\mu(b) \\ &= \frac{V(\gamma(a)}{V(\gamma(b)}E_b. \end{aligned} \tag{4.101}$$

The conclusion is that given a reference frame proportional to a Killing vector field, if an observer of the reference frame at the event x observes a particle with energy E travelling along a geodesic, then another observer of the same reference frame at the event x' observes that particle with energy E' satisfying

$$E = \frac{V(x)}{V(x')}E'. \tag{4.102}$$

We now apply this to the Unruh effect. The Rindler observers are observers of the Rindler reference frame Z_I which in turn is proportional to the Killing vector field ∂_η since the Rindler observers travel along ξ constant coordinate lines. So we may write $Z_I = V\partial_\eta$. To identify V recall Z_I must be normalized, then $g(Z_I, Z_I) = -1$. Thus we must have

$$V^2 g_{\eta\eta} = -1. \tag{4.103}$$

Since $g_{\eta\eta} = -e^{2a\xi}$ we may solve Eq. (4.103) to obtain $V(\eta,\xi) = e^{-a\xi}$.

Thus we conclude a particle with energy E at event x is observed at event x' with energy E' satisfying

$$E = e^{-a(\xi-\xi')}E'. \tag{4.104}$$

Now the observer with $\xi = 0$, which has acceleration a, observes thermal radiation with temperature T. Hence it observes a thermal energy $E = k_B T$. One observer at some arbitrary ξ' will then observe thermal energy $E' = k_B T'$ given by

$$k_B T = e^{a\xi'} E'. \tag{4.105}$$

This means the temperature the second observer will measure is given by

$$T' = e^{-a\xi'} T. \tag{4.106}$$

In that case, considering the Minkowski vacuum probed by these observers, a general Rindler observer with ξ constant will measure a temperature

$$T(\xi) = e^{-a\xi} \frac{a}{2\pi}. \tag{4.107}$$

But given the relation between ξ and the acceleration, it follows that $\alpha = ae^{-a\xi}$ is exactly the observer's proper acceleration. In that case, an uniformly accelerated Rindler observer with acceleration α will perceive the Minkowski vacuum as a thermal bath with temperature

$$T = \frac{\alpha}{2\pi}. \tag{4.108}$$

This result is what became known as the *Unruh effect*. As a final remark, notice that the origin of the effect is that the Rindler observer is causally isolated on the right Rindler wedge and cannot probe the degrees of freedom on the left Rindler wedge. This introduces one inherent mixedness on any state defined on the whole Minkowski spacetime that gets observed by the Rindler observer. This mixedness in turn reflects the uncertainty on the global state by ignorance of the complementary degrees of freedom. The Unruh effect shows that this uncertainty is sufficiently high so that the Rindler observer perceives the vacuum as a thermal state.

5

Correlation redistribution by a causal horizon

In this chapter our aim is to employ the methods of quantum information in order to establish how it is possible to quantify the correlation redistribution imparted by a causal horizon, when one of the observers of a part of the state is affected by the horizon while the other one does not, when we compare to the correlations in the case on which both observers do not perceive the horizon. We shall use some methods from the field, which became known as relativistic quantum information, to setup the problem and shall work out various correlation measures in a particular simple example.

The basic idea of the field of relativistic quantum information is to use the methods of quantum information to study correlations in quantum relativistic systems. More specifically, what is done is to consider a state of the field and discuss how observers with different notions of particles perceive the correlations. In that approach the division of a system into parts is done not by spatial division, but rather by a division in modes of a field. One considers the field as a collection of modes and considers a bipartition between modes such that one observer will observe one part of it and another observer will observe the other part. The state of the field is usually specified in terms of modes, which are appropriate for one distinguished observer - if available -, and then the part which will be probed by the observer in a distinct state of motion, is transformed accordingly following the tenets of quantum field theory in curved spacetime. It is this final form of the state that is studied from the point of view of quantum information theory. We follow closely the approach reviewed in [22]. The main issue that is investigated on this field is how one non-inertial motion, or rather how a gravitational field, affects entanglement and other correlations.

Before getting specifically into Relativistic Quantum Information, we review the methods we employed which are related to special situations for the study of correlations, regardless of the scenario - be it relativistic or not.

5.1 Special Situations for Correlations

There are special situations for correlations that should be mentioned, both of which shall be heavily used in the analysis of the next section, so that we review these here in some detail. The first situation has to do with *tripartite pure states*.

5.1.1 Tripartite pure states

In this situation suppose the Hilbert space of a quantum system is isomorphic to a triple tensor product as $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$. We call this a tripartition, and a state which has a natural description under this isomorphism is a tripartite state. The idea is the same as in the bipartite case: there is a meaningful description of a system as composed of three parts. For the purpose of discussion of correlations, the three parts can be observed in their own and the results of observations may or may not be correlated depending on the state.

It is clear that every tripartite state *can be seen as bipartite*. Simply join two parts in a single one, then the state can be seen as bipartite with respect to this decomposition. For example, define $\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$. Then $\mathscr{H} \simeq \mathscr{H}_{AB} \otimes \mathscr{H}_C$ and we describe states as bipartite with one AB part and one C part. For any tripartite state there are three bipartitions:

$$\begin{aligned} \mathscr{H} &\simeq \mathscr{H}_{AB} \otimes \mathscr{H}_C, \quad \mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B, \\ \mathscr{H} &\simeq \mathscr{H}_{AC} \otimes \mathscr{H}_B, \quad \mathscr{H}_{AC} = \mathscr{H}_A \otimes \mathscr{H}_C, \\ \mathscr{H} &\simeq \mathscr{H}_{BC} \otimes \mathscr{H}_A, \quad \mathscr{H}_{BC} = \mathscr{H}_B \otimes \mathscr{H}_C. \end{aligned} \tag{5.1}$$

Now as antecipated, the interesting case here is when the tripartite state is pure. In that case we have $\rho = |\psi\rangle\langle\psi|$ a pure and tripartite state in $\mathscr{H}$. Using Eq. (5.1) and Def. (3.2.2), which establishes the partial trace operation, we can define three bipartite states and three individual states:

$$\begin{aligned} \rho_{AB} &= \mathrm{Tr}_C\, \rho, \quad \rho_C = \mathrm{Tr}_{AB}\, \rho, \\ \rho_{AC} &= \mathrm{Tr}_B\, \rho, \quad \rho_B = \mathrm{Tr}_{AC}\, \rho, \\ \rho_{BC} &= \mathrm{Tr}_A\, \rho, \quad \rho_A = \mathrm{Tr}_{BC}\, \rho. \end{aligned} \tag{5.2}$$

The partial traces are done obviously as considering that ρ is a state on the tensor product of the appropriate two factors. These three bipartite states are appropriate to the study of observables local to AB, AC or BC, without mention to C, B or A respectively. The three individual states are the states, which are in turn appropriate to the study of observables local to A, B or C, without mention to BC, AC or AB respectively.

By the definitions it is clear that when ρ is pure, as we are assuming, we have

$$\begin{aligned} S(\rho_{AB}) &= S(\rho_C), \\ S(\rho_{AC}) &= S(\rho_B), \\ S(\rho_{BC}) &= S(\rho_A). \end{aligned} \tag{5.3}$$

Now for the bipartite states ρ_{AB}, ρ_{AC} and ρ_{BC} the mutual information will quantify the total correlations that exist between the corresponding two parts. We take notice that it simplifies greatly due to Eq. (5.3):

$$\begin{aligned} I(\rho_{AB}) &= S(\rho_A) + S(\rho_B) - S(\rho_C), \\ I(\rho_{AC}) &= S(\rho_A) + S(\rho_C) - S(\rho_B), \\ I(\rho_{CB}) &= S(\rho_C) + S(\rho_B) - S(\rho_A). \end{aligned} \tag{5.4}$$

We now turn to a result regarding the entanglement of formation and the classical correlations previously defined. To understand that, we first introduce a terminology:

Definition 5.1.1. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ and a bipartite state ρ_{AB} in $\mathscr{H}$ be given. A B-complement of ρ_{AB} is a pair $(\mathscr{H}_C, \rho_{AC})$ where $\mathscr{H}_C$ is another Hilbert space and ρ_{AC} is a state on $\mathscr{H}' \simeq \mathscr{H}_A \otimes \mathscr{H}_C$ such that there exists a tripartite pure state ρ_{ABC} on $\mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$ with the property that

$$\mathrm{Tr}_B\, \rho_{ABC} = \rho_{AC}, \quad \mathrm{Tr}_C\, \rho_{ABC} = \rho_{AB} \tag{5.5}$$

It should be clear that whenever $(\mathscr{H}_C, \rho_{AC})$ is a B-complement of ρ_{AB} then $(\mathscr{H}_B, \rho_{AB})$ is a C-complement of ρ_{AC}. With this terminology it is also clear from the previous discussions that if we start with a tripartite pure state ρ on $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$ then $(\mathscr{H}_C, \rho_{AC})$ is a B-complement of ρ_{AB} and vice versa. In the case of the tripartite pure state, since the Hilbert space used in the completion to the tripartite state is known in advance, we call the complement just ρ_{AC}, etc.

Under this kind of relation the authors of [19] prove the following theorem

Theorem 5.1.1. Let $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ and let a state ρ_{AB} be given. Suppose $(\mathscr{H}_C, \rho_{AC})$ is a B-complement of ρ_{AB}, then it holds

$$E_F(\rho_{AB}) + J^{\leftarrow}(\rho_{AC}) = S(\rho_A). \tag{5.6}$$

Thus for tripartite pure states the entanglement of formation of the reduced state of one bipartition is constrained by the classical correlations of the reduced state of a complementary bipartition.

This kind of relation can immediately also be translated into a relation with the quantum discord. For that one recalls that $J^{\leftarrow}(\rho_{AC}) = I(\rho_{AC}) - \mathcal{D}^{\leftarrow}(\rho_{AC})$ and that $I(\rho_{AC}) = S(\rho_A) + S(\rho_C) - S(\rho_B)$. This in turn gives the result that

$$E_F(\rho_{AB}) - \mathcal{D}^{\leftarrow}(\rho_{AB}) = S(\rho_C) - S(\rho_B). \tag{5.7}$$

These equations then allow that knowing the quantum discord or classical correlations for a bipartition, and the entanglement entropies of the three parts, we can recover the entanglement of formation of another bipartition.

We shall use this observation to compute the entanglement of formation in the relativistic setting. For that we shall need methods to compute the classical correlations or quantum discord. Fortunately there are such methods for a simple case.

5.1.2 Bipartite state with effective two-level system

The second special situation of interest is when we have a bipartite state ρ_{AB} with the characteristic that one of the parts is an effective two-level system. In other words, we have

$$\rho_{AB} = \sum_{i,j=0,1} \sum_{kl} \rho_{ijkl} |i\rangle_A \langle j| \otimes |k\rangle_B \langle l|, \tag{5.8}$$

so that the A part of the state is restricted to a two-dimensional subspace. When that happens we can define four operators on $\mathscr{H}_B$ by means of

$$M_{ij} = \sum_{kl} \rho_{ijkl} |k\rangle_B \langle l|, \tag{5.9}$$

so that the state becomes

$$\rho_{AB} = \sum_{i,j=0}^{1} |i\rangle\langle j| \otimes M_{ij}. \tag{5.10}$$

In that case the computation of classical correlations and quantum discord is extremely facilitated. This idea has even been used in [10] in the context of relativistic quantum information.What one observes is that *for measurements in A the system is effectively a qubit*. The simplification this entails is that any measurement will be parameterized by two angles. We show this explicitly. In a two-level system any measurement is specified by two projectors P_1 and P_2 satisfying $P_1 + P_2 = 1$. So we just call $P_1 = P$ and let $P_2 = \mathbf{1} - P$. The projector P is a hermitian operator. Since it is 2×2 it can be written in terms of Pauli matrices and the identity

$$P = \alpha \mathbf{1} + \sum_i x_i \sigma_i, \tag{5.11}$$

but now comes the projector condition. We need to have idempotency, i.e., $P^2 = P$. In that case we have

$$\left(\alpha \mathbf{1} + \sum_i x_i \sigma_i\right) \left(\alpha \mathbf{1} + \sum_j x_j \sigma_j\right) = \alpha^2 \mathbf{1} + 2\alpha \sum_i x_i \sigma_i + \sum_{ij} x_i x_j \sigma_i \sigma_j. \tag{5.12}$$

Recalling that the Pauli matrices satisfy

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i \sum_k \epsilon_{ijk} \sigma_k, \tag{5.13}$$

where ϵ_{ijk} is the Levi-Civita symbol defined so that $\epsilon_{123} = 1$ and that it is totally skew-symmetric, we can simplify the last term

$$\begin{aligned} \sum_{ij} x_i x_j \sigma_i \sigma_j &= \sum_i x_i^2 \sigma_i^2 + i \sum_{ijk} x_i x_j \epsilon_{ijk} \sigma_k \\ &= \sum_i x_i^2 \mathbf{1}, \end{aligned} \tag{5.14}$$

where in the first line we used that the last term vanishes because it involves a contraction of the symmetric $x_i x_j$ with the skew symmetric ϵ_{ijk} and in the second line we have used that $\sigma_i^2 = \mathbf{1}$. Combining the last equations the result is

$$P^2 = \left(\alpha^2 + \sum_i x_i^2 \right) \mathbf{1} + 2\alpha \sum_i x_i \sigma_i. \tag{5.15}$$

Finally equating this with P and recalling that $\{\mathbf{1}, \sigma_1, \sigma_2, \sigma_3\}$ is a basis, we get the relations

$$\alpha^2 + |\mathbf{x}|^2 = 1, \quad 2\alpha x_i = x_i. \tag{5.16}$$

The second equality implies that $\alpha = 1/2$ and the first implies that $|\mathbf{x}|^2 = 1$. Collecting the results it follows that

$$P = \frac{1}{2} \left[1 + \mathbf{x} \cdot \sigma \right], \tag{5.17}$$

where $\mathbf{x}$ is a unit vector. The complementary projector is given by

$$1 - P = \frac{1}{2} \left[1 - \mathbf{x} \cdot \sigma \right]. \tag{5.18}$$

We can call this measurement $\Pi_\pm(\mathbf{x})$ with $\Pi_+(\mathbf{x}) = P(\mathbf{x})$ and $\Pi_-(\mathbf{x}) = 1 - P(\mathbf{x})$. Now notice that what we have just proved is that: *projective measurements on a two-level system are parameterized by points on the sphere S^2, i.e, the set of all $\mathbf{x} \in \mathbb{R}^3$ with $|\mathbf{x}| = 1$.* In that setting given a bipartite state with the property that one part is an effective two-level system, it follows that the optimization problem, which defines classical correlations and quantum discord with respect to measurements on the two-level part, is in fact an optimization over two angles! Thus, even if we cannot achieve closed form expressions, for numerical computations matters are much simpler.

We work in great generality, without even considering what is the B part. We just notice that, to compute classical correlations, what we need is the state ρ_B of B prior to an arbitrary measurement and $\rho_B^\pm(\mathbf{x})$ after the measurement considering the two possible results $\pm$. Notice that the post-selected state of B depends on $\mathbf{x}$ because the measurement $\Pi_\pm(\mathbf{x})$ is specified by it. With these states in hand we are able to compute the entropies $S(\rho_B)$ and $S(\rho_B^\pm(\mathbf{x}))$ in terms of the point in the sphere $\mathbf{x} \in S^2$ characterizing the measurement. This knowledge will give rise to the function

$$J^{\rightarrow}(\rho_{AB}; \mathbf{x}) = S(\rho_B) - \sum_{\sigma=\pm} p_\sigma S(\rho_B^\sigma(\mathbf{x})) \tag{5.19}$$

of the angles, which will be optimized to give the classical correlations.

For that matter, we first compute the reduced state of B by tracing over A. This can be done simply and it yields

$$\rho_B = \mathrm{Tr}_A \sum_{ij} |i\rangle\langle j| \otimes M_{ij} = M_{00} + M_{11}. \tag{5.20}$$

The second step is to consider the measurements. Since we are measuring A we have actually that the measurements are lifted to $\mathscr{H} \simeq \mathscr{H}_A \otimes \mathscr{H}_B$ as $\Pi_\pm(\mathbf{x}) \otimes \mathbf{1}$. The post-selected states of the global system are in general written as

$$\rho^\pm(\mathbf{x}) = \frac{1}{p_\pm(\mathbf{x})}(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\rho(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1}). \tag{5.21}$$

In our case, since we only need the probabilities and the post-selected state of B it turns out that computing $(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\rho$ is already enough and since it requires less computations it is worthwhile to argue that this is really the case. Indeed, the probabilities $p_\pm(\mathbf{x})$ are given by

$$p_\pm(\mathbf{x}) = \operatorname{Tr}\left[\Pi_\pm(\mathbf{x}) \otimes \mathbf{1}\rho\Pi_\pm(\mathbf{x}) \otimes \mathbf{1}\right] \tag{5.22}$$

and by the cyclic property of the trace and the idempotency of $\Pi_\pm(\mathbf{x})$ it follows that

$$p_\pm(\mathbf{x}) = \operatorname{Tr}\left[\Pi_\pm(\mathbf{x}) \otimes \mathbf{1}\rho\right]. \tag{5.23}$$

Next we turn to the post-selected state of B. It is given by

$$\rho_B^\pm(\mathbf{x}) = \frac{1}{p_\pm(\mathbf{x})} \operatorname{Tr}_A\left[(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\rho(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\right], \tag{5.24}$$

but by Proposition (3.2.2) the partial trace is cyclic when we multiply by local operators. In that case we have

$$\rho_B^\pm(\mathbf{x}) = \frac{1}{p_\pm(\mathbf{x})} \operatorname{Tr}_A\left[(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})^2\rho\right], \tag{5.25}$$

and idempotency of $\Pi_\pm(\mathbf{x})$ again implies that this is

$$\rho_B^\pm(\mathbf{x}) = \frac{1}{p_\pm(\mathbf{x})} \operatorname{Tr}_A\left[(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\rho\right]. \tag{5.26}$$

With this we see that indeed knowing $(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\rho$ is enough. To actually compute this just need to evaluate $\Pi_\pm(\mathbf{x})|i\rangle$ with $i = 0, 1$. This turns out to be very simple by using the Pauli matrices:

$$\Pi_\pm|0\rangle = \frac{1}{2}\left(|0\rangle \pm x_1|1\rangle \pm ix_2|1\rangle \pm x_3|0\rangle\right) = \frac{1 \pm x_3}{2}|0\rangle \pm \frac{x_1 + ix_2}{2}|1\rangle, \tag{5.27a}$$

$$\Pi_\pm|1\rangle = \frac{1}{2}\left(|1\rangle \pm x_1|0\rangle \mp ix_2|0\rangle \mp x_3|1\rangle\right) = \frac{1 \mp x_3}{2}|1\rangle \pm \frac{x_1 - ix_2}{2}|0\rangle. \tag{5.27b}$$

This in turn fully characterizes how $\Pi_\pm$ acts on the basis projectors $|i\rangle\langle j|$. The resulting formulas follows immediately from the above equation by multiplying on the right by the correct bra. The result is:

$$\Pi_\pm|0\rangle\langle 0| = \frac{1 \pm x_3}{2}|0\rangle\langle 0| \pm \frac{x_1 + ix_2}{2}|1\rangle\langle 0|, \tag{5.28a}$$

$$\Pi_\pm|0\rangle\langle 1| = \frac{1 \pm x_3}{2}|0\rangle\langle 1| \pm \frac{x_1 + ix_2}{2}|1\rangle\langle 1|, \tag{5.28b}$$

$$\Pi_\pm|1\rangle\langle 0| = \frac{1 \mp x_3}{2}|1\rangle\langle 0| \pm \frac{x_1 - ix_2}{2}|0\rangle\langle 0|, \tag{5.28c}$$

$$\Pi_\pm|1\rangle\langle 1| = \frac{1 \mp x_3}{2}|1\rangle\langle 1| \pm \frac{x_1 - ix_2}{2}|0\rangle\langle 1|. \tag{5.28d}$$

With this we can compute $(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\rho$. We can write it exactly in the form of Eq. (5.10), in other words, we will have

$$(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\rho = \sum_{i,j=0}^{1} |i\rangle\langle j| \otimes \tilde{M}^\pm_{ij}, \tag{5.29}$$

with matrices

$$\tilde{M}^\pm_{00}(\mathbf{x}) = \frac{1 \pm x_3}{2}M_{00} \pm \frac{x_1 - ix_2}{2}M_{10}, \tag{5.30a}$$

$$\tilde{M}^\pm_{10}(\mathbf{x}) = \pm\frac{x_1 + ix_2}{2}M_{00} + \frac{1 \mp x_3}{2}M_{10}, \tag{5.30b}$$

$$\tilde{M}^\pm_{01}(\mathbf{x}) = \frac{1 \mp x_3}{2}M_{01} \pm \frac{x_1 - ix_2}{2}M_{11}, \tag{5.30c}$$

$$\tilde{M}^\pm_{11}(\mathbf{x}) = \pm\frac{x_1 + ix_2}{2}M_{01} + \frac{1 \mp x_3}{2}M_{11}. \tag{5.30d}$$

The probabilities are easily obtained by taking the trace. It follows that

$$p_\pm(\mathbf{x}) = \mathrm{Tr}(\Pi_\pm(\mathbf{x}) \otimes \mathbf{1})\rho = \mathrm{Tr}\, \tilde{M}_{00}^\pm + \mathrm{Tr}\, \tilde{M}_{11}^\pm, \tag{5.31}$$

or being totally explicit,

$$p_\pm(\mathbf{x}) = \frac{1 \pm x_3}{2}\,\mathrm{Tr}\, M_{00} \pm \frac{x_1 - ix_2}{2}\,\mathrm{Tr}\, M_{10} \pm \frac{x_1 + ix_2}{2}\,\mathrm{Tr}\, M_{01} + \frac{1 \mp x_3}{2}\,\mathrm{Tr}\, M_{11}. \tag{5.32}$$

These are the two probabilities. Now the post-selected states of the B system, obtained by tracing out the A system, are simply

$$\rho_B^\pm(\mathbf{x}) = \frac{1}{p_\pm(\mathbf{x})}(\tilde{M}_{00}^\pm(\mathbf{x}) + \tilde{M}_{11}^\pm(\mathbf{x})). \tag{5.33}$$

Finally, this is enough to construct the function defined in Eq. (5.19). So the problem is to maximize $J^\to(\rho_{AB};\mathbf{x})$ over the sphere.

Due to the great generality pursued here we will not be able, of course, to get a closed expression for it. Nonetheless, all of this procedure gives an idea of an algorithm to carry out this computation numerically. Although it should be clear by now, we summarize the method:

1. Define the matrices M_{ij}. If they are operators acting on some infinite-dimensional Hilbert space, we impose an upper cutoff N on the number of basis elements of $\mathscr{H}_B$ entering the definition.

2. Use Eq. (5.30a) to define the post-selected matrices $\tilde{M}_{ij}^\pm(\mathbf{x})$ as functions of $\mathbf{x} \in S^2$. With them, define $p_\pm(\mathbf{x})$ the probabilities and $\rho_B^\pm(\mathbf{x})$ the post-selected states of B.

3. Compute the entropies $S(\rho_B)$ and $S(\rho_B^\pm(\mathbf{x}))$ numerically, obviously depending on the upper cutoff N introduced on step (1). With them, define the function $J^\to(\rho_{AB};\mathbf{x})$.

4. Express $\mathbf{x} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ and maximize $J^{\rightarrow}(\rho_{AB};(x))$ in the two angles (θ, ϕ).

For the purposes of relativistic quantum information and of studying correlations in scenarios involving black holes we see two possible applications of this method:

1. In the most traditional approach to relativistic quantum information reviewed in [22] one considers that from the perspective of an inertial observer a bipartite state is prepared. The bipartition is effected with respect to modes of the field, in other words, to frequency of the possible excitations of the field, and is usually considered to have both parts as effective two-level systems. The second part is then observed from the perspective of one uniformly accelerated observer. This makes that part to become bipartite as well, as we shall review in due time. We end up with a tripartite state. In that case, the bipartitions containing the part observed by the inertial observer are bipartitions with an effective two-level system to which all of this applies. In particular, this has been carried out in [10] to study the quantum discord. Furthermore, the above discussion can be applied to the same proposal in the near horizon approximation of an eternal black hole. In that case, the inertial observer is a free-falling one, crossing the horizon in finite proper time, and the accelerated observer is a Schwarzschild observer standing still outside the black hole.

2. A proposal for discussing whether particles are observed or not by one observer consists in the introduction of the so-called Unurh-DeWitt detector [5]. It is a two-level system localized in spacetime, following a specified trajectory that couples to a quantum field and probes it. One measures the quantum field indirectly by measuring the Unruh-DeWitt detector. In particular this is capable of telling whether particles are detected or not. Since the detector is a two-level system, the methods of this section should work for it as well. In other words, given the

state of the detector plus field system at some instant of time, one may employ the methods we have outlined to compute the locally accessible and locally inaccessible information of the field to the detector. We have not, however, pursued this line of investigation in the present work.

5.2 Correlations associated to the Unruh Effect

We review the quantum information aspects associated to the Unruh effect as presented in [15, 22]. Later we shall discuss how these results can be directly transferred to a near-horizon situation in an eternal black hole spacetime, being reinterpreted in that situation. In all that follows we consider a *real* - uncharged - Klein-Gordon field, just for the sake of simplicity.

5.2.1 Description of the States

Consider a massless real Klein-Gordon field ϕ propagating in Minkowski spacetime. The inertial observer will quantize the field by expanding ϕ into modes of positive frequency according to himself. A uniformly accelerated Rindler observer, however, will quantize the field differently. The transformation between the two descriptions is done by means of the Bogoliubov transformations, as we discussed in Section 4.4. As we have also reviewed, there is a special basis of the positive frequency solutions to the inertial observer, which transform in the simplest possible way to the Rindler observer. These solutions are the so-called Unruh modes. Indeed, using the Unruh modes, the

Bogolubov coefficients that transitions to the Rindler modes are given by

$$\alpha_{ij}^{I} = \cosh \alpha_i \delta_{ij}, \tag{5.34a}$$

$$\alpha_{ij}^{II} = 0, \tag{5.34b}$$

$$\beta_{ij}^{I} = 0, \tag{5.34c}$$

$$\beta_{ij}^{II} = \sinh \alpha_i \delta_{ij}, \tag{5.34d}$$

where the α_i parameter depends on the frequency of the modes in the basis, as

$$\tanh \alpha_i = \exp\left[-\frac{\pi\omega_i}{a}\right]. \tag{5.35}$$

This in turn relates the creation and annihiliation operators of the Unruh modes $a_{i,U}, a_{i,U}^{\dagger}$ and of the Rindler modes $a_{i,I}, a_{i,I}^{\dagger}, a_{i,II}, a_{i,II}^{\dagger}$. In fact, we know from the general theory that we have the creation operator,

$$a_{i,U}^{\dagger} = \sum_j -\beta_{ij}^{I,*} a_{j,I} - \beta_{ij}^{II,*} a_{j,II} + \alpha_{ij}^{*,I} a_{j,I}^{\dagger} + \alpha_{ij}^{*,II} a_{j,II}^{\dagger}. \tag{5.36}$$

Inserting the Bogolubov coefficients given by Eq. (5.34a) we get the transformation

$$a_{i,U}^{\dagger} = -\sinh \alpha_i a_{i,II} + \cosh \alpha_i a_{i,I}^{\dagger}. \tag{5.37}$$

Now the Bogolubov transformations allow us to write the Minkowski vacuum - which is the same as the Unruh vacuum - in the Rindler basis

$$|0\rangle_j^{\Omega} = \frac{1}{\cosh \alpha} \sum_{n=0}^{\infty} \tanh^n \alpha |n\rangle_j^{I} |n\rangle_j^{II}, \quad \Omega = M, U, \tag{5.38}$$

and together with the above creation operator, we can derive the one-particle states in the Rindler system:

$$|1\rangle_j^U = \frac{1}{\cosh^2 \alpha} \sum_{n=0}^{\infty} \tanh^n \alpha \sqrt{n+1} |n+1\rangle_j^I |n\rangle_j^{II}. \tag{5.39}$$

In that setting, we consider a two-mode bipartite state, which is maximally entangled from the inertial perspective. It is also supposed that one simply looks at a two-level subsystem from the inertial perspective. For the first part we use Minkowski modes and for the second part we use Unruh modes. The two modes i and j are fixed, and all the other modes of the field are assumed to be in the vacuum state. There are many possible choices fitting these requirements, the most general one being:

$$|\phi\rangle = \sum_{a,b=0}^{1} c_{ab} |a\rangle_i^M |b\rangle_j^U, \tag{5.40}$$

where the upperscripts M and U stand for Minkowski and Unruh modes respectively. The usual choice in the literature, however, which we follow for the sake of comparison of the results, is

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_i^M |0\rangle_j^U + |1\rangle_i^M |1\rangle_j^U). \tag{5.41}$$

We are now going to derive some measures of correlations and discuss how to interpret them. Most of these are well-known from the literature [22, 15, 10] and obtaining the same results gives an important check that our computations are correct. Furthermore, using the methods of [14, 10] we find out that it is indeed possible to compute at least *one entanglement of formation*, for the bipartition between two counter-accelerating Rindler observers, an original result up to our knowledge. In the next section we shall finally understand how these measures translate into the near-horizon of an eternal black hole.

Since we are going to compute correlations, the starting point is the density operator for this state. It is

$$|\phi\rangle\langle\phi| = \frac{1}{2}\Big[|0\rangle_i^M\langle 0|\otimes|0\rangle_j^U\langle 0| + |0\rangle_i^M\langle 1|\otimes|0\rangle_j^U\langle 1| + |1\rangle_i^M\langle 0|\otimes|1\rangle_j^U\langle 0| + \\ + |1\rangle_i^M\langle 1|\otimes|1\rangle_j^U\langle 1|\Big]. \tag{5.42}$$

Notice that already at this point this fits the general formalism for composite systems on which one part is an effective two-level system. Indeed, the state given by Eq. (5.42) can be written as

$$|\phi\rangle\langle\phi| = \sum_{a,b=0}^{1} |a\rangle_i^M\langle b|\otimes M_{ab}^U, \tag{5.43}$$

where, for this particular case, M_{ab}^U are the operators

$$M_{ab}^U = |a\rangle_j^U\langle b|. \tag{5.44}$$

The Rindler observer will just observe the j-th mode. In that case we need to expand the second part of the state in the basis appropriate for her. We shall do this by simply using the Eqs. (5.38) and (5.39) applied to the mode j part in the above state. To do so, we compute the individual Unruh basis operators, from which the operators M_{ab}^U from Eq. (5.44) can be obtained by multiplying them by a factor of $1/2$:

$$|0\rangle_j^U\langle 0| = \frac{1}{\cosh^2\alpha}\sum_{n,m=0}^{\infty}\tanh^{n+m}\alpha(|n\rangle_j^I\langle m|)\otimes(|n\rangle_j^{II}\langle m|), \tag{5.45a}$$

$$|1\rangle_j^U\langle 1| = \frac{1}{\cosh^4\alpha}\sum_{n,m=0}^{\infty}\tanh^{n+m}\alpha\sqrt{n+1}\sqrt{m+1}(|n+1\rangle_j^I\langle m+1|)\otimes(|n\rangle_j^{II}\langle m|), \tag{5.45b}$$

$$|0\rangle_j^U\langle 1| = \frac{1}{\cosh^3\alpha}\sum_{n,m=0}^{\infty}\tanh^{n+m}\alpha\sqrt{m+1}(|n\rangle_j^I\langle m+1|)\otimes(|n\rangle_j^{II}\langle m|), \tag{5.45c}$$

$$|1\rangle_j^U\langle 0| = \frac{1}{\cosh^3\alpha}\sum_{n,m=0}^{\infty}\tanh^{n+m}\alpha\sqrt{n+1}(|n+1\rangle_j^I\langle m|\otimes|n\rangle_j^{II}\langle m|). \tag{5.45d}$$

All partial traces required will follow as partial traces of these operators. So we evaluate them already, since it is very simple to do. We first trace over the modes supported in II as follows:

$$\mathrm{Tr}_{II}\,|0\rangle^U_j\langle 0| = \frac{1}{\cosh^2\alpha}\sum_{n=0}^{\infty}\tanh^{2n}\alpha|n\rangle^I_j\langle n|, \tag{5.46a}$$

$$\mathrm{Tr}_{II}\,|1\rangle^U_j\langle 1| = \frac{1}{\cosh^4\alpha}\sum_{n=0}^{\infty}(n+1)\tanh^{2n}\alpha|n+1\rangle^I_j\langle n+1|, \tag{5.46b}$$

$$\mathrm{Tr}_{II}\,|0\rangle^U_j\langle 1| = \frac{1}{\cosh^3\alpha}\sum_{n=0}^{\infty}\sqrt{n+1}\tanh^{2n}\alpha|n\rangle^I_j\langle n+1|, \tag{5.46c}$$

$$\mathrm{Tr}_{II}\,|1\rangle^U_j\langle 0| = \frac{1}{\cosh^3\alpha}\sum_{n=0}^{\infty}\sqrt{n+1}\tanh^{2n}\alpha|n+1\rangle^I_j\langle n|. \tag{5.46d}$$

In the exact same way we can obtain the partial traces over the modes supported in I:

$$\mathrm{Tr}_{I}\,|0\rangle^U_j\langle 0| = \frac{1}{\cosh^2\alpha}\sum_{n=0}^{\infty}\tanh^{2n}\alpha|n\rangle^{II}_j\langle n|, \tag{5.47a}$$

$$\mathrm{Tr}_{I}\,|1\rangle^U_j\langle 1| = \frac{1}{\cosh^4\alpha}\sum_{n=0}^{\infty}(n+1)\tanh^{2n}\alpha|n\rangle^{II}_j\langle n|, \tag{5.47b}$$

$$\mathrm{Tr}_{I}\,|0\rangle^U_j\langle 1| = \frac{1}{\cosh^3\alpha}\sum_{n=0}^{\infty}\sqrt{n+1}\tanh^{2n+1}\alpha|n+1\rangle^{II}_j\langle n|), \tag{5.47c}$$

$$\mathrm{Tr}_{I}\,|1\rangle^U_j\langle 0| = \frac{1}{\cosh^3\alpha}\sum_{n=0}^{\infty}\sqrt{n+1}\tanh^{2n+1}\alpha|n\rangle^{II}_j\langle n+1|. \tag{5.47d}$$

First, we have three possible bipartitions: $\rho_{M,I}$, $\rho_{M,II}$ and finally $\rho_{I,II}$. So we wish to compute these three bipartite states. The computation is straightforward - for instance tracing over II gives by direct use of the results derived:

$$\begin{aligned}\rho_{M,I} &= \mathrm{Tr}_{II} \sum_{a,b=0}^{1} |a\rangle_i^M \langle b| \otimes M_{ab}^U = \sum_{a,b=0}^{1} |a\rangle_i^M \langle b| \otimes \mathrm{Tr}_{II}\, M_{ab}^U \\ &= \frac{1}{2\cosh^2\alpha} \sum_{n=0}^{\infty} \tanh^{2n}\alpha \left[|0n\rangle_{ij}^{MI} \langle 0n| + \frac{(n+1)}{\cosh^2\alpha} |1n+1\rangle_{ij}^{MI} \langle 1n+1| \right. \\ &\left. + \frac{\sqrt{n+1}}{\cosh\alpha} \left(|0n\rangle_{ij}^{MI} \langle 1n+1| + |1n+1\rangle_{ij}^{MI} \langle 0n| \right) \right]. \end{aligned} \tag{5.48}$$

In the same way, we trace over I, by using the same set of results

$$\begin{aligned}\rho_{M,II} &= \mathrm{Tr}_{I} \sum_{a,b=0}^{1} |a\rangle_i^M \langle b| \otimes M_{ab}^U = \sum_{a,b=0}^{1} |a\rangle_i^M \langle b| \otimes \mathrm{Tr}_{I}\, M_{ab}^U \\ &= \frac{1}{2\cosh^2\alpha} \sum_{n=0}^{\infty} \tanh^{2n}\alpha \left[|0n\rangle_{ij}^{M,II} \langle 0n| + \frac{n+1}{\cosh^2\alpha} |1n\rangle_{ij}^{M,II} \langle 1n| \right. \\ &\left. + \frac{\sqrt{n+1}\tanh\alpha}{\cosh\alpha} \left(|0n+1\rangle_{ij}^{M,II} \langle 1n| + |1n\rangle_{ij}^{M,II} \langle 0n+1| \right) \right]. \end{aligned} \tag{5.49}$$

Finally we trace over M instead, and this yields

$$\begin{aligned}\rho_{I,II} &= \mathrm{Tr}_{M} \sum_{a,b=0}^{1} |a\rangle_i^M \langle b| \otimes M_{ab}^U = \sum_{a,b=0}^{1} \mathrm{Tr}(|a\rangle_i^M \langle b|) M_{ab}^U = M_{00}^U + M_{11}^U \\ &= \frac{1}{2\cosh^2\alpha} \sum_{n,m=0}^{\infty} \tanh^{n+m}\alpha \left(|nn\rangle_j^{I,II} \langle mm| + \frac{\sqrt{n+1}\sqrt{m+1}}{\cosh^2\alpha} |n+1n\rangle_j^{I,II} \langle m+1m| \right). \end{aligned} \tag{5.50}$$

Eqs. (5.48), (5.49) and (5.50) are the possible bipartite states we can extract from the tripartite state, respectively the state probed by the Minkowski observer and the Rindler observer in the right Rindler wedge, by the Minkowski observer and the Rindler observer in the left Rindler wedge, and by the two Rindler observers. The next step is to derive the states of the individual parts M, I, II probed respectively by the Minkowski observer, the Rindler observer on the right Rindler wedge and the Rindler observer on the left Rindler wedge. To obtain ρ_I and ρ_{II} it is easier to trace $\rho_{I,II}$. For ρ_I we get

$$\begin{aligned}\rho_I &= \mathrm{Tr}_{II}\,\rho_{I,II}\\ &= \mathrm{Tr}_{II}\,M^U_{00} + \mathrm{Tr}_{II}\,M^U_{11}.\end{aligned} \tag{5.51}$$

These two partial traces have already been computed in Eqs. (5.46a) and (5.46b), collecting the results gives

$$\rho_I = \frac{1}{2\cosh^2\alpha}\sum_{n=0}^{\infty}\tanh^{2n}\alpha\left(|n\rangle^I_j\langle n| + \frac{n+1}{\cosh^2\alpha}|n+1\rangle^I_j\langle n+1|\right), \tag{5.52}$$

which can be simplified by substituting $m = n+1$ in the second sum. Effecting this change yields the form of the state:

$$\rho_I = \sum_{n=0}^{\infty}\frac{\tanh^{2(n-1)}\alpha}{2\cosh^2\alpha}\left(\tanh^2\alpha + \frac{n}{\cosh^2\alpha}\right)|n\rangle^I_j\langle n|. \tag{5.53}$$

The same procedure yields ρ_{II}. We trace out I getting

$$\begin{aligned}\rho_{II} &= \mathrm{Tr}_I\,\rho_{I,II}\\ &= \mathrm{Tr}_I\,M^U_{00} + \mathrm{Tr}_I\,M^U_{11}.\end{aligned} \tag{5.54}$$

We again have already computed these partial traces in Eqs. (5.47a) and (5.47b), so that

$$\rho_{II} = \frac{1}{2\cosh^2\alpha}\sum_{n=0}^{\infty}\tanh^{2n}\alpha\left(1 + \frac{n+1}{\cosh^2\alpha}\right)|n\rangle^{II}_j\langle n|. \tag{5.55}$$

Finally the individual state ρ_M is the simpler to obtain. We can simply trace the I, II part to get

$$\mathrm{Tr}\,|\phi\rangle\langle\phi| = \sum_{a,b=0}^{1}|a\rangle^M_i\langle b|\,\mathrm{Tr}\,M^U_{ab}. \tag{5.56}$$

One can then easily see that the matrices of the operators M^U_{01} and M^U_{10} have no diagonal elements and hence have zero trace. One thus finds

$$\mathrm{Tr}\,|\phi\rangle\langle\phi| = \left(\sum_{n=0}^{\infty}\frac{\tanh^{2n}\alpha}{2\cosh^2\alpha}\right)|0\rangle\langle 0| + \left(\sum_{n=0}^{\infty}\frac{(n+1)\tanh^{2n}\alpha}{2\cosh^4\alpha}\right)|1\rangle\langle 1|. \tag{5.57}$$

These sums can be computed explicitly and both of them have value $1/2$. This means that

$$\rho_M = \frac{1}{2}(|0\rangle^M_i\langle 0| + |1\rangle^M_i\langle 1|). \tag{5.58}$$

This was in fact expected and could be also computed by using the initial form of the state, before converting the second part to the Rindler basis.

5.2.2 Entropies and Mutual Information

The mutual information can be computed directly using the methods of our review in Subsection 5.1.2. In particular, recall that if we have a tripartite pure state ρ_{ABC} then considering two parts together to be a single subsystem allows us to view it as a bipartite pure state, e.g. $\rho_{(AB)C}$ where we have grouped A and B together in a single subsystem. Because of that, since it is pure, we have that $S(\rho_{AB}) = S(\rho_C)$, also $S(\rho_{AC}) = S(\rho_B)$ and finally $S(\rho_{CB}) = S(\rho_A)$. Therefore it turns out that it suffices to know the entropies of the individual parts. This also implies, from the definition of the mutual information, that we have

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_C), \tag{5.59a}$$

$$I(\rho_{AC}) = S(\rho_A) + S(\rho_C) - S(\rho_B), \tag{5.59b}$$

$$I(\rho_{CB}) = S(\rho_C) + S(\rho_B) - S(\rho_A). \tag{5.59c}$$

This is exactly the case at hand. Actually this is quite general for the study of what happens to correlations when one passes to the Rindler frame. The reason is that we start with a bipartite state from the perspective of the inertial reference frame and then suppose that one of the parts will actually be observed by a Rindler observer. In that case that part must be transformed and that ends up adjoining the left Rindler wedge inevitably, so we always end up with a tripartite pure state in the setting considered here.

The situation is even more favorable to the analysis because the reduced density matrices ρ_M, ρ_I and ρ_{II} are all diagonal, which allows us to read of the eigenvalues which enter the computation of the von-Neumman entropy. In fact, the ρ_M one is trivial and has entropy $S(\rho_M) = 1$. The other two must be handled by numeric methods.

We first plot the entropies of the three individual states, all against the parameter α which contains the information about the acceleration.:

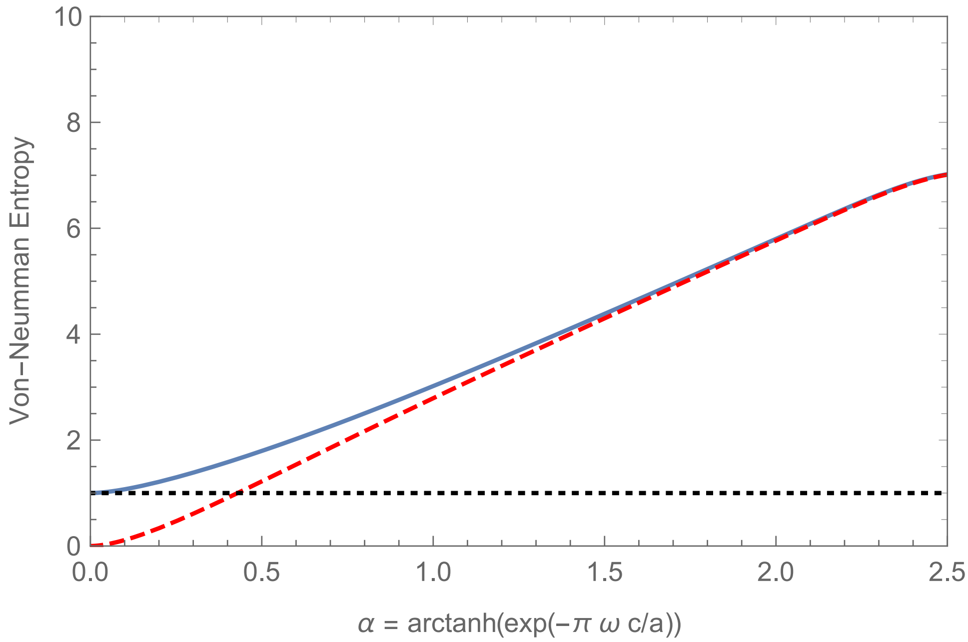

FIGURE 5.1: Von-Neumman entropies of the states $\rho_M, \rho_I, \rho_{II}$. The state ρ_M is shown by the black (dotted) line, ρ_I by the blue (solid) line and ρ_{II} by the red (dashed) line.

The entropies, as we previously explained in Chapter 2, quantify the mixedness of a state. What the above shows is that if the inertial observer sees a two-mode pure bipartite state, which is maximally entangled as an effective two-level system state, then two counter-accelerating observers will see a mixed state. Furthermore the mixedness increases almost linearly with $\alpha = \text{arctanh}(\exp(-\pi\omega c/a))$ where ω is the frequency of the mode observed by the accelerating observers and a is their acceleration which differs just in direction for the two of them. Therefore acceleration increases the uncertainty about the state.

Now to better understand the correlations we plot the mutual information for the three bipartitions. Since these are obtained by just simple algebraic combinations of the entropies, this is easily obtained from the entropies plotted above:

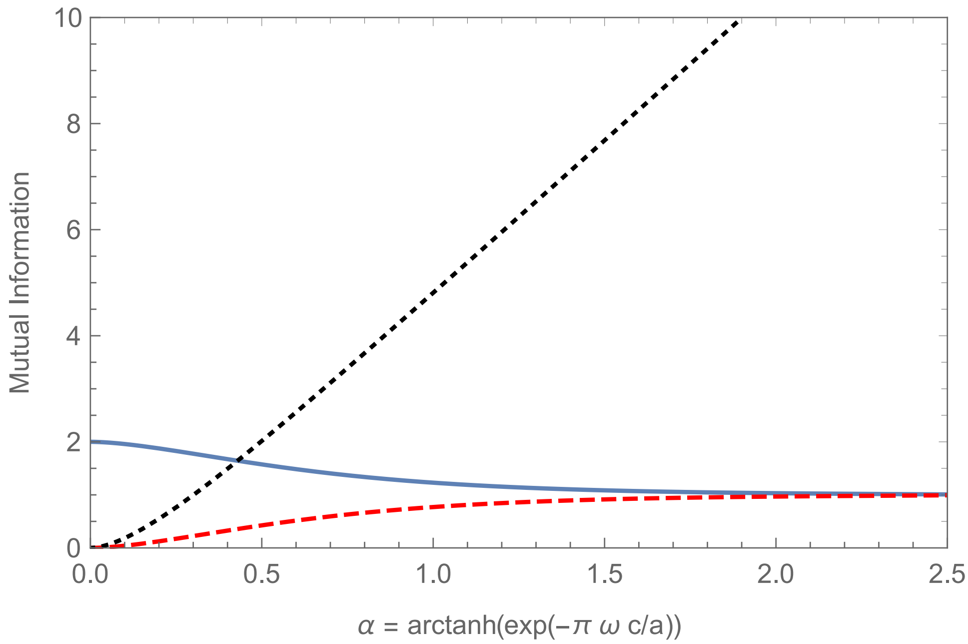

FIGURE 5.2: Mutual information - The state ρ_{MI} is the blue solid line, ρ_{MII} is the red dashed line and $\rho_{I,II}$ is the black dotted line.

In Fig. (5.2) we see that the higher the squeeze parameter the more the correlations of the bipartition between the Minkowski and Rindler observer on the right Rindler wedge decrease and the more the correlations between the Minkowski and Rindler observer on the left Rindler wedge increase. Moreover, this follows a conservation law. In fact, we have from Eq. (5.59a):

$$I(\rho_{M,I}) + I(\rho_{M,II}) = [S(\rho_M) + S(\rho_I) - S(\rho_{II})] + [S(\rho_M) + S(\rho_{II}) - S(\rho_I)] = 2S(\rho_M), \tag{5.60}$$

and upon recalling that $S(\rho_M) = 1$ we find

$$I(\rho_{M,I}) + I(\rho_{M,II}) = 2, \tag{5.61}$$

which works as a conservation law which is obeyed as a correlation transfer occurs from the bipartition $\rho_{M,I}$ to the bipartition $\rho_{M,II}$.

5.2.3 Classical Correlations and Quantum Discord

We now set out to find out the classical correlations and quantum discord. As it has been already pointed out, this is in general a complicated optimization problem with no general method to be applied. Still, following the method of [10], which we extensively reviewed in greater generality in 5.1.2, we can take advantage of the fact that the ρ_M part is an effective two-level system. In other words, the M part behaves as a qubit, and hence measurements made on ρ_M can be parameterized by two angles.

This discussion applies to the bipartitions MI and MII which have an effective two-level part. Let us consider the MI bipartition. The appropriate operators defining the state are

$$M_{00}^{MI} = \sum_{n=0}^{\infty} \frac{\tanh^{2n}\alpha}{2\cosh^2\alpha} |n\rangle\langle n|, \tag{5.62a}$$

$$M_{01}^{MI} = \sum_{n=0}^{\infty} \frac{\sqrt{n+1}\tanh^{2n}\alpha}{2\cosh^3\alpha} |n\rangle\langle n+1|, \tag{5.62b}$$

$$M_{10}^{MI} = \sum_{n=0}^{\infty} \frac{\sqrt{n+1}\tanh^{2n}\alpha}{2\cosh^3\alpha} |n+1\rangle\langle n|, \tag{5.62c}$$

$$M_{11}^{MI} = \sum_{n=0}^{\infty} \frac{(n+1)\tanh^{2n}\alpha}{2\cosh^4\alpha} |n+1\rangle\langle n+1|. \tag{5.62d}$$

These are the prior to measurement operators. They are the only input we need, the remaining of the process has been carried out in Subsection 5.1.2 and written entirely in terms of M_{ab}^{MI}. We plot, in Fig. (5.3), the classical correlation obtained, together with the mutual information, containing the full correlations for this bipartite state, and together with the quantum discord which is easily obtainable from the other two:

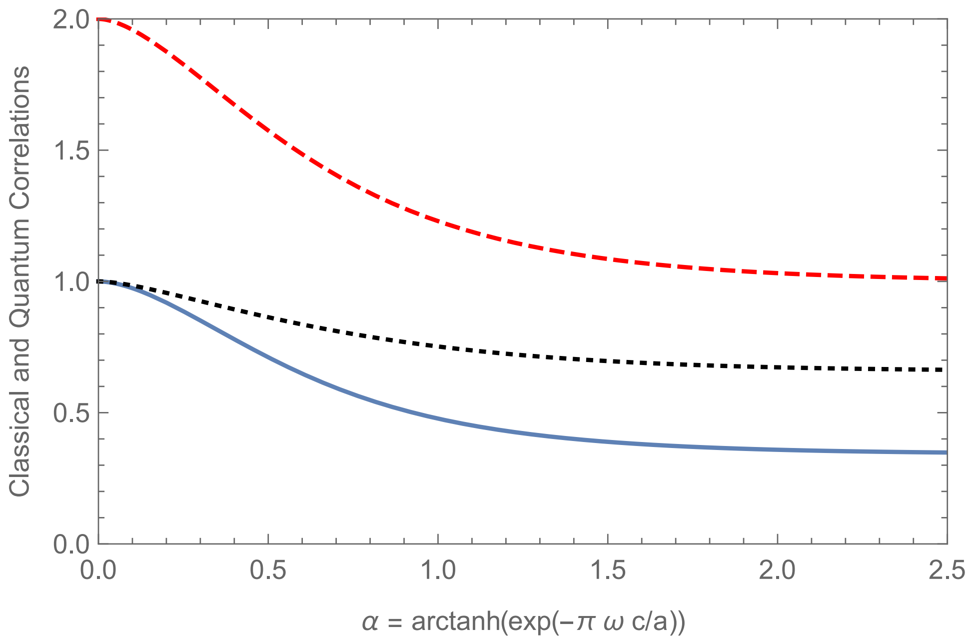

FIGURE 5.3: State ρ_{MI} - Classical Correlations is the solid blue line, quantum discord is the black (dotted) line and mutual information is the red (dashed) line.

Another bipartition for which the method applies is the MII bipartition. In that case, the appropriate matrices are

$$M_{00}^{MII} = \sum_{n=0}^{\infty} \frac{\tanh^{2n}\alpha}{2\cosh^2\alpha}|n\rangle\langle n|, \tag{5.63a}$$
$$M_{01}^{MII} = \sum_{n=0}^{\infty} \frac{\sqrt{n+1}\tanh^{2n+1}\alpha}{2\cosh^3\alpha}|n+1\rangle\langle n|, \tag{5.63b}$$
$$M_{10}^{MII} = \sum_{n=0}^{\infty} \frac{\sqrt{n+1}\tanh^{2n+1}\alpha}{2\cosh^3\alpha}|n\rangle\langle n+1|, \tag{5.63c}$$
$$M_{11}^{MII} = \sum_{n=0}^{\infty} \frac{(n+1)\tanh^{2n}\alpha}{2\cosh^4\alpha}|n+1\rangle\langle n+1|. \tag{5.63d}$$

The exact same procedure can be employed in this case. With it we get a second classical correlation, whose plot compared to the mutual information and quantum discord of the bipartition is shown in Fig. (5.4).

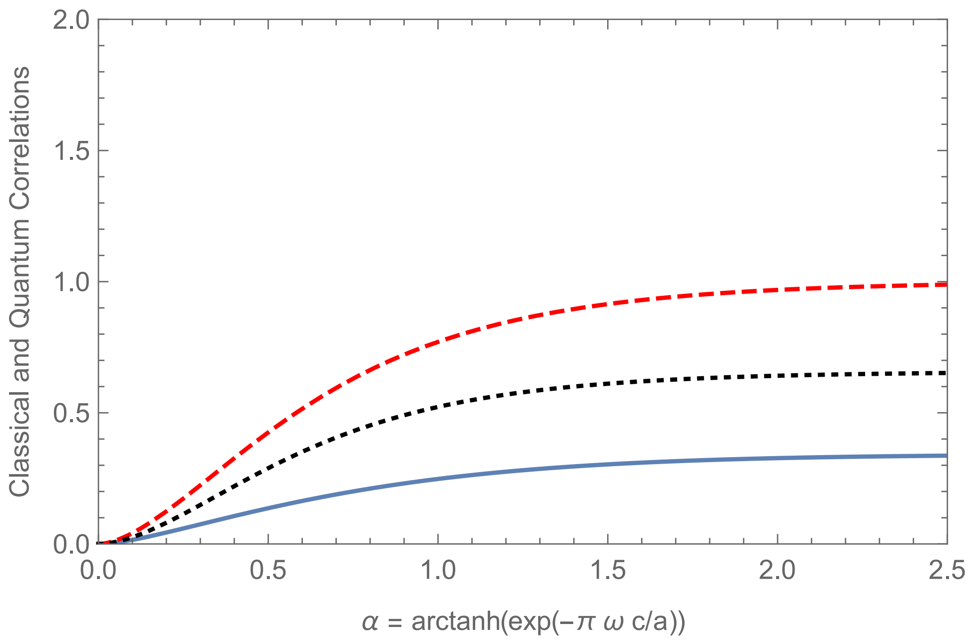

FIGURE 5.4: State ρ_{MII} - Classical Correlations is the solid blue line, quantum discord is the black dotted line and mutual information is the red dashed line.

Finally, to discuss the results, it is very instructive to plot all correlations measures (classical correlations, quantum discord and mutual information) for the two bipartitions together. Doing so, using different colors for each bipartition we obtain the plot shown in Fig. (5.5).

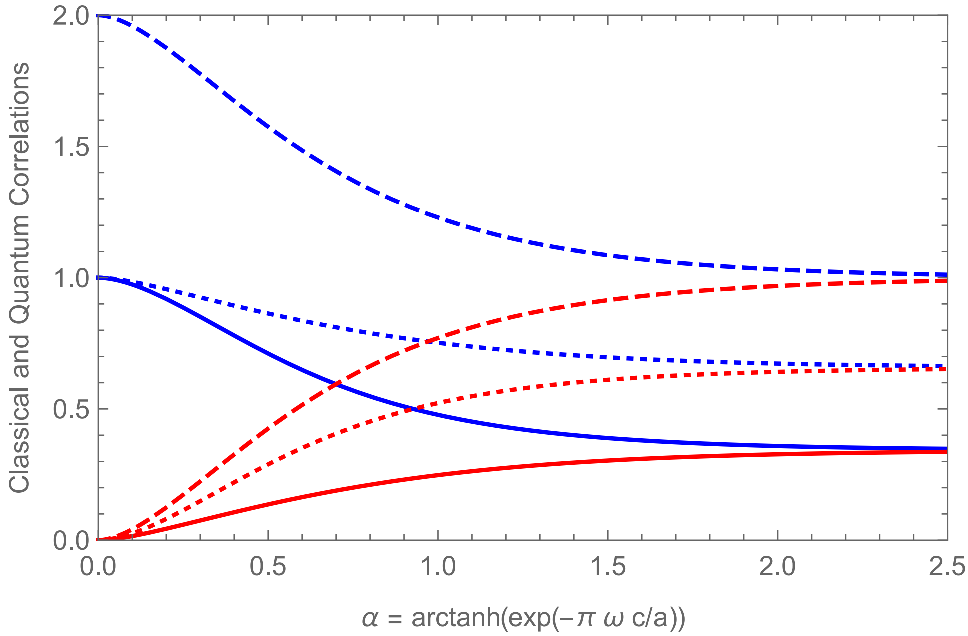

FIGURE 5.5: States $\rho_{M,I}$ and $\rho_{M,II}$ compared - The state $\rho_{M,I}$ is depicted by the blue lines and $\rho_{M,II}$ be the red lines. Classical Correlations are the solid lines, quantum discord are the dotted lines and mutual information are dashed lines.

In Fig. (5.5), the blue lines are the plots for the bipartition between the inertial observer and the right Rindler observer, with the classical correlations and quantum discord characterizing respectively the locally accessible and locally inacessible information for the inertial observer. The red lines are the plots for the bipartition among the inertial observer and the left Rindler observer and now the classical correlations and quantum discord characterize respectively the locally accessible and locally inacessible information for the inertial observer. The case of zero acceleration and hence zero squeeze parameter is obviously the case in which we are considering just inertial observers. Hence we clearly see in the plot that when there is a non-zero acceleration, compared to the situation in which there is not, a tradeoff of the correlations occur.

5.2.4 Entanglement of Formation

We now turn to the discussion of entanglement of formation of the bipartition among the two Rindler observers, which, following [19], can be interpreted as signaling the correlation redistribution we have seem in the previous section. Our methodology is to use the methods for tripartite pure states reviewed in 5.1.1 and read the entanglement of formation out of the classical correlation which can be computed, even if numerically. The only drawback of the method, is that since measurements must be done in the effective two-level part, we are only able to obtain the entanglement of formation for one bipartition. Still it is of significance since it signals the correlation redistribution.

Different from the negativity that has been computed earlier in the literature [15, 22], the entanglement of formation is a very hard measure to compute in practice, unless for special cases. The problem is the optimization which directly enters its definition. Still, in this special case, we can use the correlations we found to compute *at least one entanglement of formation*.

In fact, since we know $J^{\rightarrow}(\rho_{MI})$ we can use the equation

$$J^{\rightarrow}(\rho_{MI}) + E_F(\rho_{I,II}) = S(\rho_I), \tag{5.64}$$

which upon being inverted yields the entanglement of formation of $\rho_{I,II}$. In fact we get

$$E_F(\rho_{I,II}) = S(\rho_I) - J^{\rightarrow}(\rho_{MI}). \tag{5.65}$$

Using this equation, we plot the entanglement of formation of the state $\rho_{I,II}$, probed by the two counter-accelerating Rindler observers, against the parameter α characterizing the acceleration. The result is shown in Fig. (5.6). In fact, in [23, 22] the negativity for

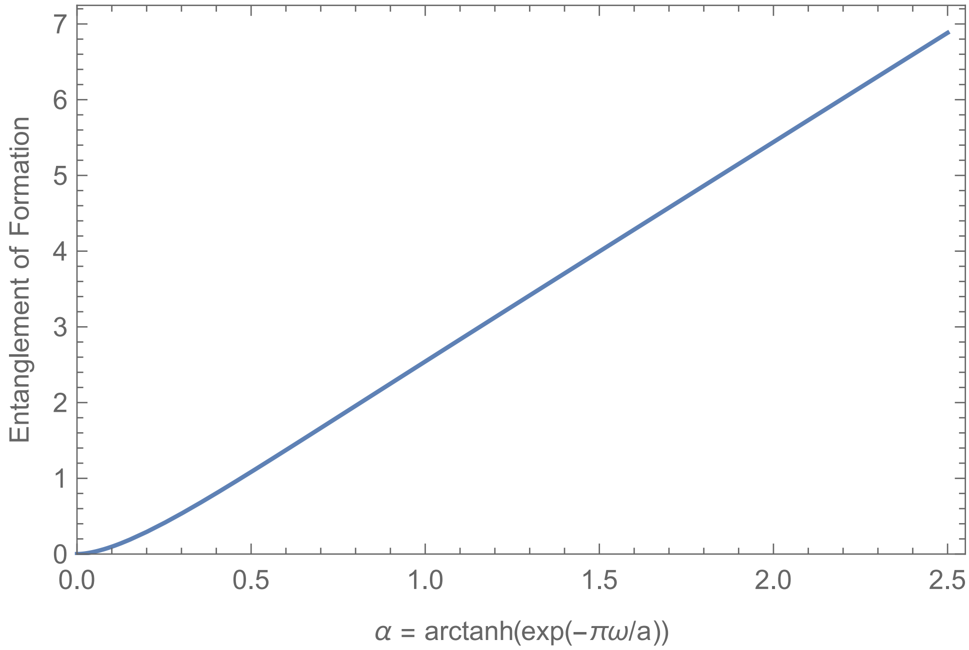

FIGURE 5.6: State $\rho_{I,II}$ - Entanglement of formation

this bipartition was computed and plotted against the squeeze parameter. The author's plot is shown in Fig. (5.7) for comparisson. We call attention that the parts of the state we have called I and II, in reference to the Rindler regions, the author has called Rob and AntiRob respectively, giving names to the corresponding observers. Comparing their plot, of negativity, with ours, of entanglement of formation, we see that the overall behavior of the obtained entanglement of formation matches, at least qualitatively, that of the negativity.

5.3 Analysis near the horizon of an eternal Black Hole

Interestingly the results obtained can be immediately adapted to the near-horizon region of an eternal Schwarzschild black hole. This happens as a result of the fact that Rindler spacetime is a near-horizon approximation to the Schwarzschild spacetime [35, 22], a fact we reviewed in Section 2.4.

We recall that in the near-horizon region the right Schwarzschild exterior region

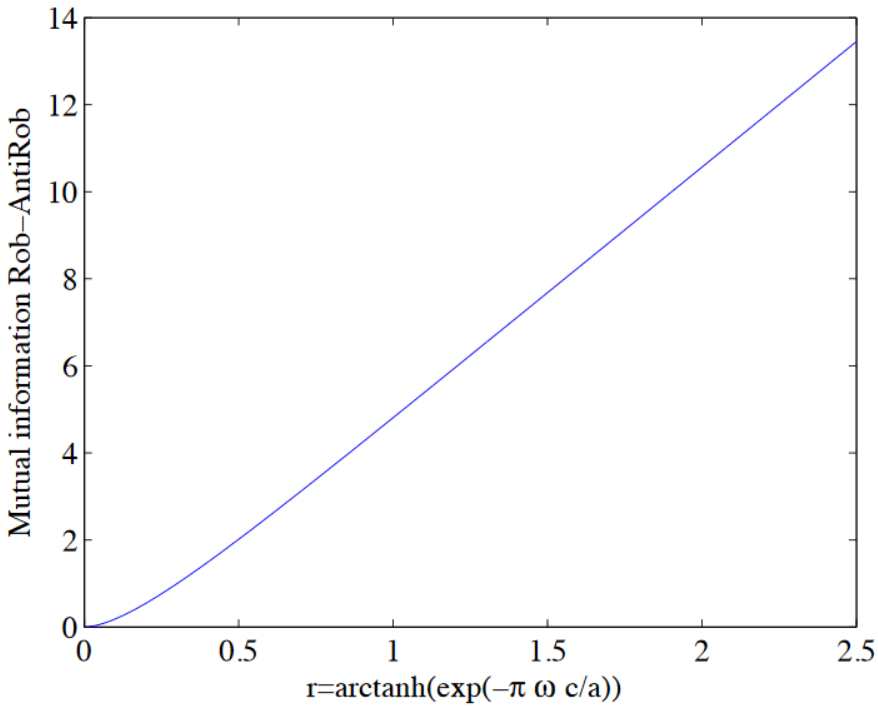

FIGURE 5.7: Plot from [22, 23] showing the negativity of the state probed by the two counteraccelerating Rindler observers. The author has called the right Rindler observer Rob and the left Rindler observer AntiRob. We see that the qualitative behavior of this entanglement measure matches our entanglement of formation from Fig. (5.6)

has its geometry well approximated by the right Rindler geometry. Similarly, the left Schwarzschild exterior region has its geometry well approximated by the left Rindler geometry. In that situation, a Schwarzschild observer corresponds to a Rindler observer, while a free-falling observer corresponds to a Minkowski observer. In that case the Minkowski vacuum and associated Fock space picture corresponds to the construction of a quantum theory based on the so-called Hartle-Hawking vacuum state, appropriate for the free-falling observer. The Rindler vacuum and its associated Fock space picture, corresponds to the construction of a quantum theory based on the so-called Boulware vacuum state, appropriate for the Schwarzschild observer.

The Boulware construction has a natural bipartition into the two regions, deprived of classical communication. In fact, local observables on any of the two regions cannot be probed by the other, and any observer supported in either of them needs to trace out the degrees of freedom supported in the complementary region, which for himself lies *beyond the horizon*.

The results of the preceeding section carry over directly to this new situation with just a new squeeze parameter,

$$\tanh\alpha = \exp\left[-\frac{\pi\omega}{\kappa}\sqrt{f(r_0)}\right], \tag{5.66}$$

or by using $\kappa = 1/4M$ and $f(r_0) = 1 - 2M/r_0$,

$$\tanh\alpha = \exp\left[-4M\pi\omega\sqrt{1-\frac{2M}{r_0}}\right]. \tag{5.67}$$

In fact, by defining $\Omega = 8M\pi\omega$ and $R_0 = r_0/2M$ we are able to get a simplified squeeze parameter,

$$\tanh\alpha = \exp\left[-\frac{\Omega}{2}\sqrt{1-\frac{1}{R_0}}\right]. \tag{5.68}$$

In this situation Ω has the meaning of the frequency of the mode observed by the Schwarzschild observer when seen by himself and R_0 his radial positioning in units of Schwarzschild radius [22, 23]. In that case, the mutual information, classical correlations and quantum discord for the bipartition corresponding to the free-falling observer and to the Schwarzschild one, which is the analogue of the $M - I$ bipartition in the Rindler case, are all plotted for $\Omega = 1$ as function of R_0 in Fig. (5.8).

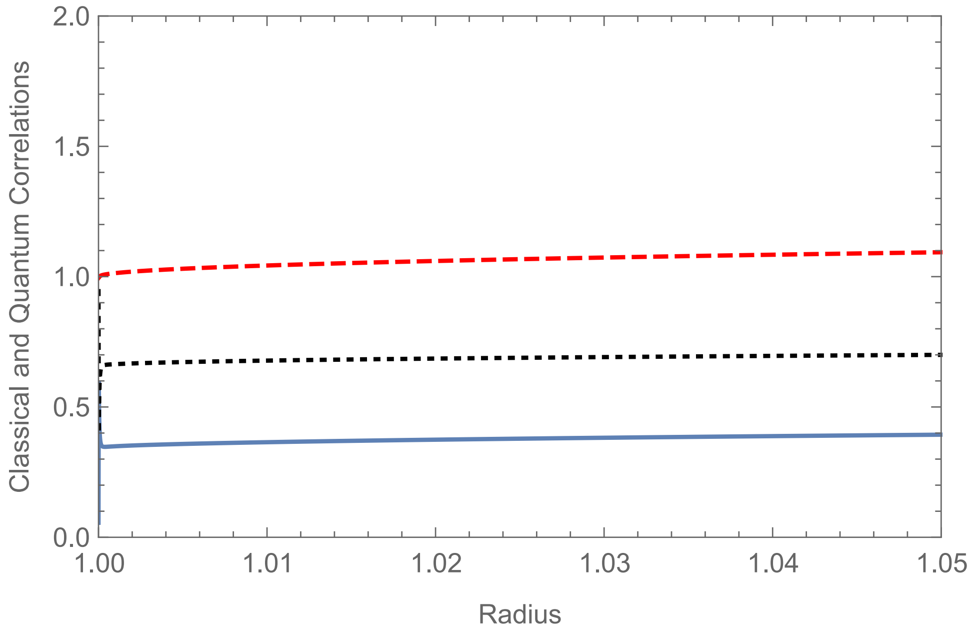

FIGURE 5.8: State $\rho_{H,B}$ - Classical Correlations is the solid blue line, quantum discord is the black dotted line and mutual information is the red dashed line.

We call attention to the fact that the plot is restricted to a very small range of R_0, since in order that the results can be used we need to ensure the near-horizon approximation holds.

In the same way, we can also plot the correlations for the bipartition between the free-falling observer and the Schwarzschild observer sitting on the parallel exterior region, beyond the horizon. The result we get is shown in Fig. (5.9).

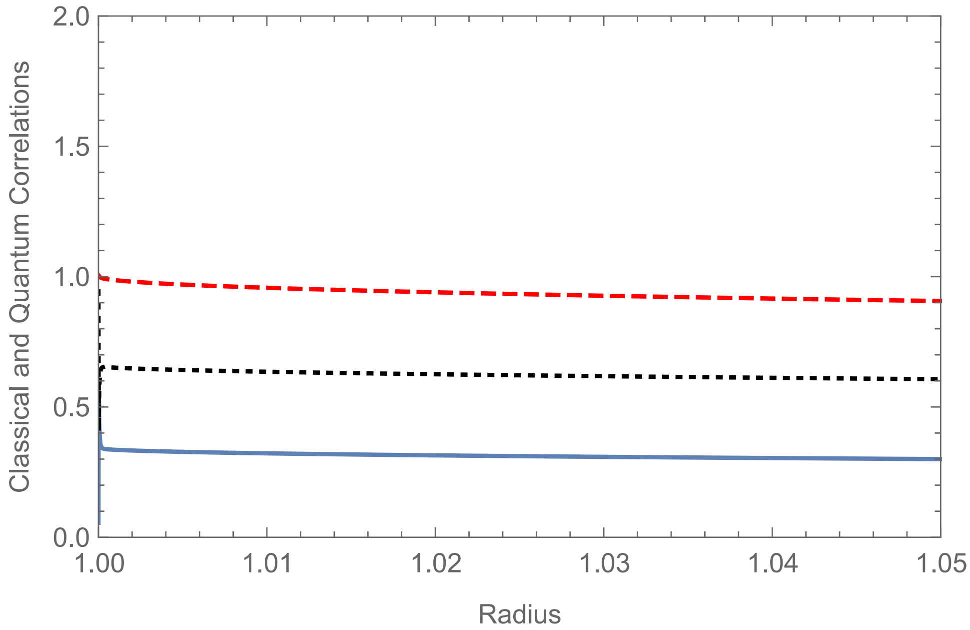

FIGURE 5.9: State $\rho_{H,\bar{B}}$ - Classical Correlations is the solid blue line, quantum discord is the black dotted line and mutual information is the red dashed line.

As we did in the Rindler situation, we can plot all correlations together to better see the correlation tradeoff. The plot is shown in Fig. (5.10), and shows that the fact that the Schwarzschild observers experience the horizon makes the correlations be redistributed compared to the situation in which all observers of the system are in free-fall. We again see all correlations of the bipartition between the free-falling and right Schwarzschild observer decrease and the correlations of the bipartition between the free-falling and left Schwarzschild observer increase, which shows the correlation redistribution.

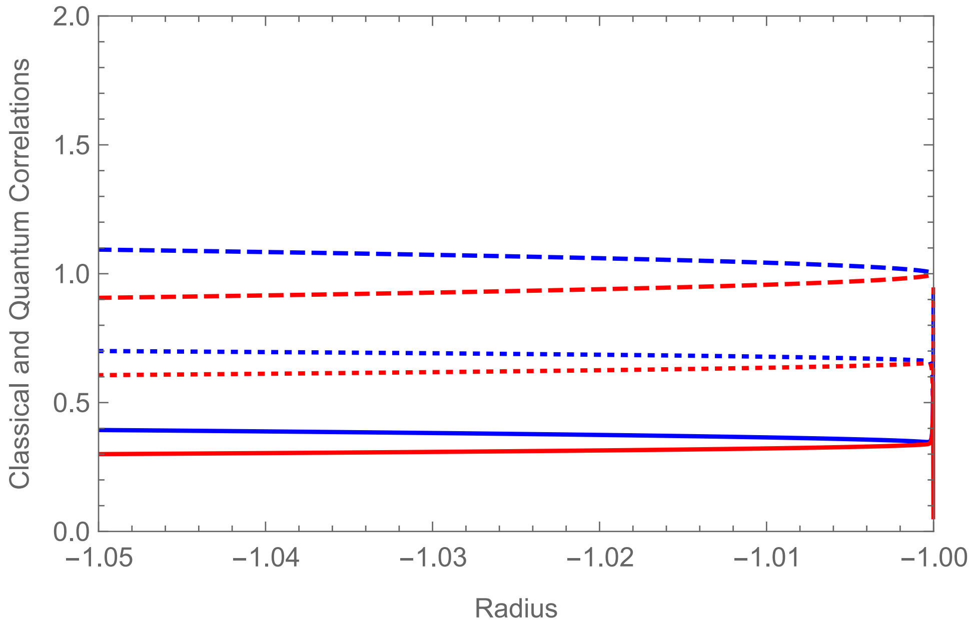

FIGURE 5.10: States $\rho_{H,B}$ and $\rho_{H,\bar{B}}$ compared - The state $\rho_{H,B}$ is depicted by the blue lines and $\rho_{H,\bar{B}}$ by the red lines. Classical Correlations are the solid lines, quantum discord are the dotted lines and mutual information are dashed lines.

Finally the knowledge of classical correlations allows us to use the methods for tripartite pure states in order to write down the entanglement of formation for the bipartition between the two Schwarzschild observers on each side of the horizon. The entanglement of formation of the state seen by these two observers is given in Fig. (5.11).

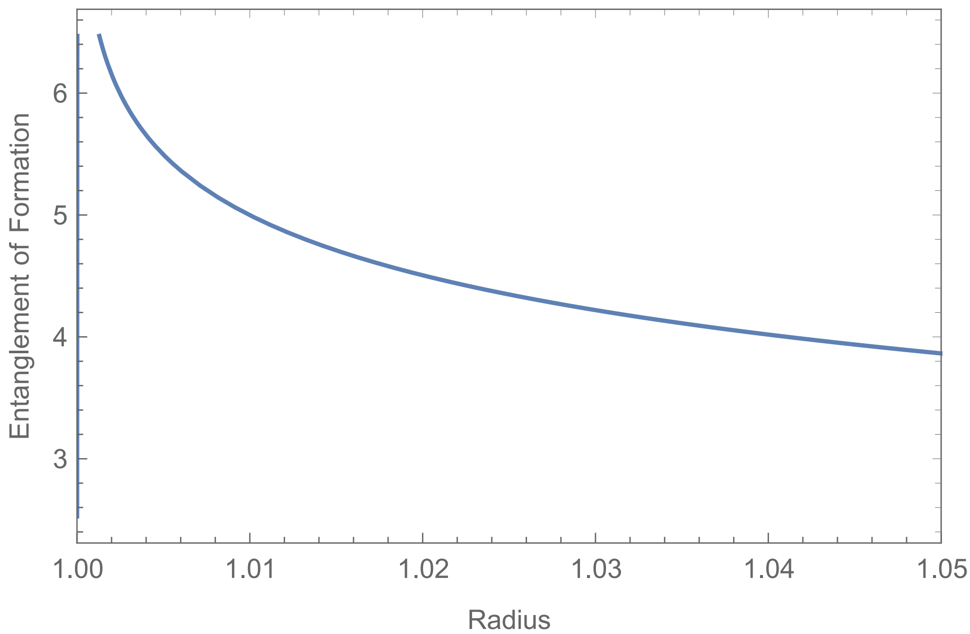

FIGURE 5.11: State $\rho_{B,\bar{B}}$ - Entanglement of formation

In fact, this entanglement is not "useful" for quantum information processing as a quantum resource, since in order that it be useful one would also need classical communication between the observers of the bipartition, which the causal structure of Schwarzschild does not allow [22, 23]. Nevertheless, the fact that the plotted measure is the *entanglement of formation*, instead of negativity, allows this entanglement to be interpreted in terms of the classical correlation and the available information for the other bipartitions for which the classical communication *is allowed*.

In fact, we wish to interpret this in the light of the relation

$$E_F(\rho_{B,\bar{B}}) = S(\rho_B) - J^{\rightarrow}(\rho_{H,B}). \tag{5.69}$$

In fact, $J^{\rightarrow}(\rho_{H,B})$ tells the maximum reduction of uncertainty in the state of B which can be effected by measurements in H. Obviously the maximum value it can attain is $S(\rho_B)$, corresponding to the case where full information about B can be, *in principle*, accessible to H. Now, the fact is that $S(\rho_B) - J^{\rightarrow}(\rho_{H,B})$ being greater than zero means that the accessible information is less that the total information needed to fully specify B. This means that the entanglement of formation of $\rho_{H,B}$ appears as the signature of information about B being not accessible to measurements at H and the bigger it is, more information is inaccessible.

In fact, the plots show that as close as the observer is to the horizon, the bigger the entanglement of formation gets to the bipartition $B, \bar{B}$. In fact this means that the closer the Schwarzschild observer is to the horizon, the more the horizon will make the information about the free-falling observer inaccessible and this will reflect in the appearence of a bigger entanglement of formation. This seems compatible with the well-known decoherence effect that the presence of a horizon generates to the Schwarzschild observer, which in effect even renders the vacuum thermal to himself as we have seen in the Unruh effect. In that sense, even though in fact the bipartition $B, \bar{B}$ is deprived of classical communication rendering the entanglement not useful as a resource, the fact that one is able to measure the entanglement of formation is a useful tool for interpretation, since it has direct meaning as a measure of inaccessible information as we have tried to argue here.

5.4 Conclusions

We have constructed two very simple examples, the first considering the presence of the Rindler horizon and the second using this to transfer the results to the near-horizon region of a maximally extended Schwarzschild geometry. In both examples we first considered that a real scalar Klein-Gordon field was observed by two observers, both of which do not perceive a causal horizon. For the Rindler situation, those would be two Minkowski observers whereas for the Schwarzschild situation, those would be two free-falling observers. In that situation, we supposed that the field was in a very simple bipartite state with respect to the two observers, with known correlations.

In particular the state we considered was a maximally entangled two-mode state on which at most one excitation on each mode could be detected.

We then considered that the same state postulated is observed now by replacing one of the observers with another one, which perceives the causal horizon. For the Rindler case this means considering that one of the observers is replaced by a Rindler observer, with specific uniform acceleration through his whole existence, whereas for the Schwarzschild case this means considering that one of the observers is replaced by a Schwarzschild observer, standing still outside of the black hole at a certain radial coordinate. In this situation, we studied how the correlations changed when compared to the simpler first situation. This should give an idea of the effect of the causal horizon on the correlations of a quantum state.

The main feature of the analysis of the horizons we considered is that the transformation from the basis appropriate to the notion of particles of the observer who does not perceive the horizon to the basis appropriate to the notion of particles of the observer who perceives the horizon introduces one third part beyond the horizon observed by a parallel observer who also perceives a horizon.

More concretely, in the Minkowski case, the transformation from Minkowski to Rindler basis introduces a left Rindler wedge part of the state, which could be observed by a Rindler observer supported on that region but not by a Rindler observer supported on the right Rindler wedge. In the Schwarzschild case, the transformation from the Hartle-Hawking basis appropriate to the free-falling observer to the Boulware basis appropriate to the Schwarzschild observer introduces a parallel exterior region part of the state, which this time could be observed by a Schwarzschild observer supported on the parallel universe but not by a Schwarzschild observer on the usual exterior region.

This means that comparing the two situations - no horizon perceived against horizon perceived by one of the parts - a state on which in the first situation is naturally bipartite, in the second is naturally tripartite. This opens up a rich possibility of correlation redistribution between the bipartite states that can be extracted from this tripartite state.

In fact for the Minkowski case we plotted $J^{\rightarrow}(\rho_{M,I})$ and $\mathcal{D}^{\rightarrow}(\rho_{M,I})$ as well as $J^{\rightarrow}(\rho_{M,II})$ and $\mathcal{D}^{\rightarrow}(\rho_{M,II})$ which are measures of locally accessible and locally inaccessible information contained in the correlations between the Minkowski observer and the Rindler observer supported in regions I and II against the parameter characterizing the acceleration of the Rindler observer.

6

Conclusions and Final Comments

We finally present the conclusions of this work, with comments on the limitations of the method employed and with some remarks on the possibility of future work on this line of investigation. First regarding the setup, we considered a very simple example that has been under consideration in the relativistic quantum information community for some time now [15, 22, 10].

To recall, considering always first the Rindler case, the simple example we mention is the example on which two modes of a scalar Klein-Gordon field are observed first by two inertial observers, in that case being well-described by the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_i|0\rangle_j + |1\rangle_i|1\rangle_j), \tag{6.1}$$

and then is observed by an inertial observer and a Rindler observer. In that particular case this bipartite state is in fact mixed, as the Rindler observer does not have complete access to the state the second inertial observer had. In fact, in order for this Rindler observer to really describe exactly what the two inertial observers would observe, he also needs the part on the opposite Rindler wedge observed by a secondary Rindler observer. Since this is impossible as the two observers are separated by the causal structure of spacetime there is an inherent mixedness introduced due to the lack of information.

Comparing the expression of the state as bipartite accross two inertial observers and tripartite accross one inertial and two Rindler observers, one is able to understand the effect imparted by the causal horizon on the distribution of information among the parts. Computing the classical, quantum and total correlations for the bipartite states probed by the inertial observer and one of the Rindler observers, discarding the part of the state probed by the second Rindler observer, we can see the information redistribution, which gets larger as one is closer to the horizon. Information is clearly lost, as one would qualitatively expect, and it is signaled by the entanglement between the two parts probed by the two Rindler observers.

The results does not seem to depend on the form of the state we assumed for the inertial observer. In fact, it seems that there are two factors which are the ones of importance here:

- The subsystem that will be probed by the Rindler observer, when transformed to the appropriate Rindler basis becomes a tensor product. Thus the state of the system is bipartite for two inertial observers but tripartite for one inetial and two Rindler observers to convey the same physics. This bipartite to tripartite transition seems one of the biggest responsibles for the result;

- The fact that Minkowski particle states are highly entangled for the perspective of two Rindler observers. Allied with the first point, this is responsible for a high degree of entanglement between the Rindler parts. In turn, this implies redistribution of correlations for the other bipartitions, signaling transfer of information from one bipartition to the other;

- The causal horizon makes one of the Rindler parts be denied access to the second Rindler part. This creates one inherent mixedness in the state probed by a Rindler observer. In fact, a Rindler observer does not discard the part of the state associated to the complementary Rindler wedge simply because he decides to not observe such degrees of freedom, as may often happen. In the present case, the Rindler observer is *denied* knowledge of these degrees of freedom.

The method employed, however, has limitations. First the method relied heavily on the fact that the inertial part of the state was a two-level state. If it was not the case, the parameterization of the measurements that allowed the numeric optimization to be carried out could not be done. Thus, for more general states we encounter the usual problem in quantum information theory that the optimizations required for actual computations of correlation measures to be very hard.

The method also does not allow for the computation of correlations for all bipartitions. Just for two bipartitions, we are able to compute classical correlations and quantum discord, since they have a two-level part. The other bipartition has both parts with states which in the natural number basis has all coefficients non-zero. For these ones computing classical correlations and quantum discord is extremely hard. In the same line, the method just allows for the computation of a single entanglement of formation. The measure of entanglement is even harder to compute in general and using its definition it seems almost hopeless to be able to do it. In that case to compute it we rely on the correlations measures which by the above remarks cannot be done for the other bipartition, denying the computation of more entanglement measures. Still, bearing aside the limitations of the method, which are simply reflections of the overall difficulty encountered in quantum information theory to compute the optimizations required for correlations and entanglement measures, for the reasons we have listed, we believe the conclusion to be general: a causal horizon redistributes correlations when compared to the situation on which it is not present.

Furthermore, in the process of trying to quantify this correlation redistribution we found out a method to compute entanglement of formation of a mixed state which admits a purification to a tripartite state which has a two-level subsystem. This is the result of direct combination of the methods of [10] which allows for the computation of classical correlations and quantum discord by optimizing over S^2 when a state has a two-level part, with the methods of [19] which relate entanglement of formation of a bipartition to the classical correlation of a complementary bipartition. This method, although still involves an optimization, it is often much easier than the one that defines entanglement of formation. Organizing this method seems to be a relevant conclusion from the point of view of quantum information theory *per se*.

Finally we draw some speculative comments on future work. One interesting line of inquiry could be to consider the so-called Unruh-DeWitt detectors. These are two-level systems, so they could replace the inertial part of the state. In that case restricting to a two-level part would seem more natural. Considering the detector coupled to a quantum field, the methods used here could allow for an understanding of how the information contained in correlations of the field may become accessible to the detector, and mainly, how much such information is accessible given the detector parameters. We would expect, in particular, a dependence on the range of energies the detector is sensible to.

Another important line of inquiry would be to consider one gravitational collapse black hole. In particular in this scenario we have the Hawking evaporation mechanism which implies the black hole in fact *emits* particles and in turn gradually evaporates. A recent proposal due to Strominger [34] is that in this process there is, in addition to the Hawking quanta, a companion creation of soft gravitons due to accelerations of the outgoing Hawking quanta. This proposal is justified either by requiring one infrared finite transition amplitude or by conservation of BMS charges as argued by Strominger.

In that case, if one considers the full quantum state of matter plus the gravitational perturbation, as soon as the event horizon forms and the collapsing matter lies inside of it, one ends up with a bipartite state accross the event horizon that acts like one entangling surface. On the exterior region, one could still consider a bipartition between hard and soft modes. It could be interesting to formulate and compute the information redistribution in this process. It seems, however, that this would require methods which go far beyond the ones considered here.

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Appendices

A

Lorentzian Geometry

In this appendix we review the basics of Lorentzian geometry. This material can be found in many references, e.g. [31, 29, 24]. First of all, we define the signature of a metric tensor.

Definition A.1.1. Let V be a vector space over $\mathbb{R}$. A metric tensor on V is a symmetric bilinear map $g : V \times V \to \mathbb{R}$. We shall say that g has signature (p, q), with $p + q = \dim V$, if there is a basis $\{e_i\}$ on which its matrix representation assumes the diagonal form with entries being -1 p times and $+1$ q times.

In our convention, the entries -1 on the diagonal of the metric tensor matrix will come first and then the entries equal to $+1$. There are two cases of signatures which are important to be given names.

Definition A.1.2. Let V be a vector space over $\mathbb{R}$ and g a metric tensor on V. If the signature of g is $(0, \dim V)$ we say that g is a Riemmanian metric tensor, while if its signature is $(1, \dim V - 1)$ we say that g is Lorentzian.

A metric tensor on V gives rise, in the same way that an ordinary inner product, to an isomorphism between V and its dual V^* called the *musical isomorphism*. This is done by taking $v \in V$ and mapping to the covector $v^\flat : V \to \mathbb{R}$ by means of $v^\flat(w) = g(v, w)$. This $\flat : V \to V^*$ being a linear isomorphism has one inverse $\sharp : V^* \to V$. We shall call $v^\flat$ the physically equivalent covector or one-form to v. In the same sense we call $\omega^\sharp$ the physically equivalent vector to ω. This identification allows one to define the so-called inverse metric, which is a metric tensor on V^*. It is defined by $g^\sharp : V^* \times V^* \to \mathbb{R}$ by means of $g^\sharp(\omega, \eta) = g(\omega^\sharp, \eta^\sharp)$.

We see that a Riemannian metric tensor is actually just one inner product. Also, since as a vector space V is always isomorphic to $\mathbb{R}^{\dim V}$, we can focus basically just on this case. We shall refer to the pair $(\mathbb{R}^n, g)$ where g is a metric tensor of signature (p, q) by $\mathbb{R}^{p,q}$. In particular, we can thus define the Minkowski vector space alluded to in the historical introduction

Definition A.1.3. We shall call $\mathbb{R}^{1,n-1}$ the n-dimensional Minkowski vector space.

The four-dimensional Minkowski vector space is the space of all spacetime intervals considered in Special Relativity. Its elements comprise all separation of events, and hence, all directions in spacetime. This continues to be true in General Relativity, although, it holds just infinitesimally. In that case, at each point one element of the associated Minkowski vector space points to infinitesimally nearby events. In regard to these separations, they are classified according to the definition:

Definition A.1.4. Let $\mathbb{R}^{1,n-1}$ be the n-dimensional Minkowski vector space. We say that $v \in \mathbb{R}^{1,n-1}$ is timelike if $g(v, v) < 0$, spacelike if $g(v, v) > 0$ or lightlike if $g(v, v) = 0$. Likewise we say that $\eta \in {\mathbb{R}^{1,n-1}}^*$ is timelike if $g^\sharp(\eta, \eta) < 0$, spacelike if $g^\sharp(\eta, \eta) > 0$ or lightlike if $g^\sharp(\eta, \eta) = 0$.

Notice that, by definition, in the covector case, η is timelike, spacelike or lightlike if and only if $\eta^\sharp$ is timelike, spacelike or lightlike. The terminology holds because when

$v \in \mathbb{R}^{1,n-1}$ is timelike, it points towards a possibly "pure time direction". In other words, a massive particle could be headed along v in spacetime. When v is spacelike it points towards a possibly "pure spatial direction", whereas when v is lightlike it points towards a direction on which massless particles could be going.

We now turn to Lorentzian manifolds, the case on which all of the above still holds locally:

Definition A.1.5. A Lorentzian manifold of dimension n is a pair (M, g) where M is a smooth n-dimensional manifold and $g \in \sec T^0_2(M)$ is a $(0,2)$ tensor field, which associates to each $p \in M$ a Lorentzian metric tensor $g_p : T_pM \times T_pM \to \mathbb{R}$.

Remark that due to the above definition, for each $p \in M$ the tangent space T_pM is actually isomorphic to the Minkowski vector space. We now briefly recall then that a linear connexion ∇ on the tangent bundle TM of the spacetime manifold is said metric compatible if $\nabla g = 0$. In that setting, we have the fundamental theorem of Riemmanian geometry, adapted to the Lorentzian case:

Theorem A.1.1. Let (M, g) be a Lorentzian manifold, then there is a unique metric compatible and torsion free connexion ∇ on TM.

The connexion is locally specified by the so-called connexion coefficients. Recall that these are defined as follows: let $X, Y \in \sec TM$ be two vector fields. We wish to compute $\nabla_X Y$ in a coordinate chart (U, x). We first expand $X = X^\mu \partial_\mu$ and $Y = Y^\nu \partial_\nu$. By using the properties of the connexion it follows

$$\begin{aligned}
\nabla_X Y &= \nabla_{X^\mu \partial_\mu} Y^\nu \partial_\nu \\
&= X^\mu [\nabla_\mu Y^\nu \partial_\nu] \\
&= X^\mu [(\nabla_\mu Y^\nu)\partial_\nu + Y^\nu \nabla_\mu \partial_\nu]
\end{aligned} \tag{A.1}$$

expanding $\nabla_\mu \partial_\nu = \Gamma^\alpha_{\mu\nu}\partial_\alpha$, renaming the indices and factoring the basis vector yields the result:

$$\nabla_X Y = [X^\mu \partial_\mu Y^\alpha + X^\mu Y^\alpha \Gamma^\alpha_{\mu\nu}]\partial_\alpha. \tag{A.2}$$

The first term of Eq. (A.2) is just the componentwise action of X on Y. The second term is what actually encodes the connexion. The connexion coefficients are thus defined by

$$\nabla_\mu \partial_\nu = \Gamma^\alpha_{\mu\nu}\partial_\alpha. \tag{A.3}$$

The connexion ∇ on TM allows us to define what we mean by the autoparallel curves of the connexion. These are the curves $\gamma : [a,b] \to M$ satisfying the equation

$$(\gamma^* \nabla)_{\frac{d}{ds}} \gamma' = 0. \tag{A.4}$$

On the other hand, we can talk about the geodesics of the metric g. These are, by definition, the curves $\gamma : [a,b] \to M$ which are extreme points of the energy functional defined by

$$E(\gamma) = \frac{1}{2}\int_a^b g_{\gamma(t)}(\gamma'(t), \gamma'(t))dt. \tag{A.5}$$

We recall that metric compatibility means that the autoparallel curves of the metric compatible connexion ∇ are exactly the geodesics of the metric g. The geodesic equation can be found exactly as the Euler-Lagrange equations for the above functional. In particular, the Lagrangian of interest is

$$L = \frac{1}{2} g_{\gamma(t)}(\gamma'(t), \gamma'(t)). \tag{A.6}$$

This equation can be derived in local coordinates. For that one works in a coordinate chart (U, x) with coordinate functions x^μ. In the tangent bundle one naturally has coordinates x^μ and $\dot{x}^\mu$ with the property that if a vector $v \in TM$ lives at the point $p \in M$ in the coordinate basis it reads

$$v = \dot{x}^\mu(v) \frac{\partial}{\partial x^\mu}\bigg|_p. \tag{A.7}$$

In that setting one can express L in terms of x^μ and $\dot{x}^\mu$ coordinates and work out the Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu}. \tag{A.8}$$

In particular, it is shown in differential geometry texts [33] as well as in general relativity ones [7, 39] that this procedure yields the equations

$$\ddot{\gamma}^\mu(\lambda) + \Gamma^\mu_{\alpha\beta}(\gamma(\lambda))\dot{\gamma}^\alpha(\lambda)\dot{\gamma}^\beta(\lambda) = 0. \tag{A.9}$$

This reveals a method to compute more efficiently the connexion coefficients. One simply derives the Euler-Lagrange equations for the lagrangian given by Eq. A.6 and reads the terms according to the above result.

Since the geodesic equation is a differential equation its solutions are functions. Hence what one gets by solving the geodesic equation is *a parameterization of a curve* $\gamma : I \subset \mathbb{R} \to M$. If we reparameterize the curve with a function $\xi : J \to I$ to get a new curve $\tilde{\gamma} = \gamma \circ \xi$ then in general it won't satisfy the same differential equation, even though the geometric image of the curve is the same. Substituting $\tilde{\gamma} \circ \xi$ on the geodesic equation, it is straightforward to find that upon invoking that γ satisfies the geodesic equation, $\tilde{\gamma}$ satisfies

$$\ddot{\tilde{\gamma}}^\mu + \Gamma^\mu_{\alpha\beta}\dot{\tilde{\gamma}}^\alpha\dot{\tilde{\gamma}}^\beta = \xi''\dot{\tilde{\gamma}}^\mu, \tag{A.10}$$

so we see that $\tilde{\gamma}$ satisfies the geodesic equation if and only if $\xi'' = 0$ and hence if and only if

$$\xi(\lambda) = a\lambda + b. \tag{A.11}$$

Therefore if a curve satisfies the geodesic equation, the set of curves with same image satisfying the equation as well are related to the first one by *affine reparameterizations*. The parameter of a curve satisfying the geodesic equation is called *an affine parameter*. An arbitrary parameter for a curve need not have any intrinsic meaning, but the affine parameter has. In the case of a spacelike or timelike curve, the affine parameter is the proper length or proper time along the curve. So it has one intrinsic geometric significance [39, 7]. In fact, even though a lightlike curve has zero length as measured by the Lorentzian metric, the affine parameter is a natural and geometrical evolution parameter along it that can be seen as a generalization of "length" for said curves.

Because the affine parameter is geometrically meaningful, instead of being one arbitrary parameter, the following definition is well motivated:

Definition A.1.6. Let (M, g) be a Lorentzian manifold and $\gamma : I \subset \mathbb{R} \to M$ an affinely parameterized geodesic. We call γ *complete* if $I = \mathbb{R}$, otherwise we call it *incomplete*.

Notice that this definition is well posed exactly because we have one distinguished kind of parameterization. In fact, in general a curve could be defined just on a small interval because of a bad parameterization. Take for instance the curve in the plane $\gamma : (-\pi/2, \pi/2) \to \mathbb{R}^2$ defined by $\gamma(\lambda) = (\lambda, \lambda^2)$. Reparameterizing the curve sending $\lambda \mapsto \tan\theta$ we get a curve defined on all of $\mathbb{R}$. But since we are requiring that the condition holds for an affinely parameterized geodesic, no allowed reparameterization will make one incomplete geodesic complete. In fact affine reparameterizations will take a bounded parameter interval to another bounded one.

We next turn to a useful classification of functions and coordinates on M.

Definition A.1.7. Let (M, g) be a Lorentzian manifold and $f \in C^\infty(M)$ a smooth function. We say that f is timelike, spacelike or lightlike on the open set $U \subset M$ if the covector field df is timelike, spacelike or lightlike respectively on each point of U. When $U = M$ we just call df timelike, spacelike or lightlike.

Now recall that a coordinate system (x, U) on spacetime has associated coordinate functions $x^\mu : U \to \mathbb{R}$. For fixed μ we say that x^μ is a timelike, spacelike or lightlike coordinate according to the corresponding classification of the smooth function x^μ on U.

We now recall the idea of a symmetry of a Lorentzian manifold, that would be a geometry-preserving map:

Definition A.1.8. Let (M, g) be a Lorentzian manifold. An *isometry* is one diffeomorphism $\phi : M \to M$ with the property that $\phi^* g = g$. In other words

$$(\phi^* g)_q(X_q, Y_q) = g(\phi(q))(X_{\phi(q)}, Y_{\phi(q)}), \quad \forall q \in M, X_q, Y_q \in T_q M$$

Next, recall that if X is a vector field on a manifold M we get from X a one-parameter family of diffeomorphisms, called the flow of X and usually denoted Φ_t^X for $t \in (-\epsilon, \epsilon)$ with the property that $\Phi_t^X(q)$ moves q along the integral line of X by a parameter value t. Conversely, if Φ_t is a one-parameter family of diffeomorphisms, we can always define

$$X_q = \left.\frac{d}{dt}\right|_{t=0} \Phi_t(q). \tag{A.12}$$

This is well defined because for q fixed, $t \mapsto \Phi_t(q)$ is a differentiable curve which can be differentiated at $t = 0$ to yield a tangent vector living in $T_q M$. This procedures defines a vector field which clearly yields $\Phi_t^X = \Phi_t$.

Thus one-parameter families of diffeomorphisms and vector fields are in bijective correspondence. The vector field corresponding to a one-parameter family of diffeomorphisms is called its *generator*. It turns out that when Φ_t^K is a one-parameter family of isometries in the sense above described, the generator K satisfies one special equation [7, 39]

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{A.13}$$

This is called Killing's equation and K is called a *Killing vector field*.

B

Globally Hyperbolic Spacetimes

B.1 Causal relation between events

Here we review aspects of the the causal structure of a spacetime and in the particular the notion of a Cauchy surface which underlies the definition of the class of globally hyperbolic spacetimes, which is the main class of spacetimes which are used as background in QFT. This begins with the definitions bellow:

Definition B.1.1. Let (M, g) be a spacetime and $\mathfrak{e} \in M$ an event. We say that $\mathfrak{e}' \in M$ chronologically precedes $\mathfrak{e}$ when there is a past-directed timelike curve $\gamma : [0,1] \to M$ with $\gamma(0) = \mathfrak{e}$ and $\gamma(1) = \mathfrak{e}'$ and we write $\mathfrak{e}' \prec \mathfrak{e}$. We say that $\mathfrak{e}'$ chronologically succeeds $\mathfrak{e}$ when $\mathfrak{e}$ chronologically preceeds $\mathfrak{e}'$. We further define

$$\begin{aligned} I^-(\mathfrak{e}) &= \{\mathfrak{e}' \in M : \mathfrak{e}' \prec \mathfrak{e}\}, \\ I^+(\mathfrak{e}) &= \{\mathfrak{e}' \in M : \mathfrak{e} \prec \mathfrak{e}'\} \end{aligned} \tag{B.1}$$

which are respectively called the chronological past and future of $\mathfrak{e}$.

An event $\mathfrak{e}'$ chronologically precedes $\mathfrak{e}$ when an observer can be at $\mathfrak{e}'$ and on his future be at $\mathfrak{e}$. In particular, such observer, using his propertime, can assign a time lapse between the events. The same intuition holds for an event chronologically succeding another. Another notion is given by the *causal past or future*.

Definition B.1.2. Let (M, g) be a spacetime and $\mathfrak{e} \in M$ an event. We say that $\mathfrak{e}' \in M$ causally precedes $\mathfrak{e}$ when there is a past-directed causal - i.e. timelike or lightlike - curve $\gamma : [0, 1] \to M$ with $\gamma(0) = \mathfrak{e}$ and $\gamma(1) = \mathfrak{e}'$ and we write $\mathfrak{e}' \prec \mathfrak{e}$. We say that $\mathfrak{e}'$ causally succeeds $\mathfrak{e}$ when $\mathfrak{e}$ causally preceeds $\mathfrak{e}'$. We further define

$$\begin{aligned} J^-(\mathfrak{e}) &= \{\mathfrak{e}' \in M : \mathfrak{e}' \prec \mathfrak{e}\}, \\ J^+(\mathfrak{e}) &= \{\mathfrak{e}' \in M : \mathfrak{e} \prec \mathfrak{e}'\}, \end{aligned} \tag{B.2}$$

which are respectively called the causal past and future of $\mathfrak{e}$.

The intuition here is that we are also considering events so that light can travel between them. Putting it more simply, the causal past of an event is the set of all events which could have causal influence upon it and the causal future is the set of all events which it could influence causally.

We now review, following closely [39] the notion of extendibility of a causal curve. To make this precise we define what is an endpoint of the curve:

Definition B.1.3. Let (M, g) be a spacetime and let $\gamma : I \subset \mathbb{R} \to M$ be a future directed causal curve. We say that $p \in M$ is a future endpoint of γ if for every neighborhood $U \subset M$ of p there is t_0 such that for all $t > t_0$ we have $\gamma(t) \in U$. Similarly we say that $q \in M$ is a past endpoint of γ if for every neighborhood $V \subset M$ of q there is t_0 such that for all $t < t_0$ we have $\gamma(t) \in V$.

The idea of future endpoint is that we can get arbitrarily close to it just being sufficiently "to the future" along the curve. The past endpoint follows the same idea. As pointed out in [39], if a future endpoint exists it is unique because every spacetime is a Hausdorff topological space. The proof is very succinct, and we reproduce it here. Suppose there were two such endpoints p and q with $p \neq q$, then we could find two disjoint open sets U_p and U_q with $p \in U_p$ and $q \in U_q$. The definition implies that there are t_p and t_q such that for all $t > t_p$ we have $\gamma(t) \in U_p$ and for all $t > t_q$ we have $\gamma(t) \in U_q$. Thus for all $t > \max\{t_p, t_q\}$ we have $\gamma(t) \in U_p \cap U_q$ which is a contradiction since this intersection is empty by the Hausdorff property. Thus, the future endpoint is unique. The same thing holds for a past endpoint, the proof being the same. With this in hands we finally define inextendibility:

Definition B.1.4. Let (M, g) be a spacetime and let $\gamma : I \subset \mathbb{R} \to M$ be a future-directed causal curve. We say it is future inextendible if it has no future endpoint. Similarly it is past inextendible if it has no past endpoint.

Now let $S \subset M$ be a set of events. The notion of inextendibility allows us to talk about the set of events which are entirely determined by S. Now we ask, when one event $p \in M$ is fully specified by the happening in S? To answer that consider all future directed, past inextendible causal curves which pass through p. If all of those intersect S, we may say that whatever happens at p is fully determined by S. This leads to:

Definition B.1.5. Let (M, g) be a spacetime and $S \subset M$ a closed achronal set. We define the *future domain of dependence* of S to be the set of all events such that every past inextendible causal curve passing through them intersect S. The future domain of dependence is denoted $D^+(S)$. Similarly the *past domain of dependence* of S is the set of all events such that every future-inextendible causal curve passing through them intersects S and is denoted $D^-(S)$. The domain of dependence is $D(S) = D^-(S) \cup S \cup D^+(S)$.

The reason for the technical conditions of being closed and achronal is discussed in [39]. The basic idea is that the future domain of dependence is the set of all events with the property that whatever happens at them is completely characterized by the events in S. The past domain of dependence is the set of events which fully characterize what happens in S.

We finally define what is a Cauchy surface, which is the correct definition of "initial value locus" in general relativity:

Definition B.1.6. Let (M, g) be a spacetime. A Cauchy surface is $\Sigma \subset M$ with the property that $D(\Sigma) = M$. When there's a Cauchy surface, we say that (M, g) is *globally hyperbolic*.

B.2 Foliation by Cauchy Surfaces

We mention that there's a very important result which states that when (M, g) is globally hyperbolic, it can actually be foliated by Cauchy surfaces. In other words, it is $M \simeq \mathbb{R} \times \Sigma$ where Σ is a Cauchy surface. Furthermore, each $\Sigma_t = \{t\} \times \Sigma$ is a Cauchy surface itself.

To be more precise, the result is that there is an isometry $\varphi : \mathbb{R} \times \Sigma \to M$ so that one foliates M with the surfaces $\Sigma_t \subset M$ defined by $\varphi(\{t\} \times \Sigma)$. The foliation gives rise on M to two datum. The first is a time function. We define $t : M \to \mathbb{R}$ to be $t(\mathfrak{e}) = \lambda$ when $\mathfrak{e} \in \Sigma_\lambda$. This $t \in C^\infty(M)$ because it is in reality $t = \mathrm{pr}_1 \circ \varphi^{-1}$ where pr_1 is the projection onto the first factor of $\mathbb{R} \times \Sigma$ which is smooth in the same way as φ^{-1}. Secondly, the foliation defines a time vector field, $\mathfrak{T}$. To define it, pick $\mathfrak{e} \in M$. Since φ is bijective, there is $(\lambda_\mathfrak{e}, q_\mathfrak{e}) \in \mathbb{R} \times \Sigma$ with $\varphi(\lambda_\mathfrak{e}, q_\mathfrak{e}) = \mathfrak{e}$. Fixing $q_\mathfrak{e}$ we get a curve which passes through $\mathfrak{e}$ for parameter value $\lambda_\mathfrak{e}$. Define

$$\mathfrak{T}_\mathfrak{e} = \left.\frac{d}{d\lambda}\right|_{\lambda_\mathfrak{e}} \varphi(\lambda, q_\mathfrak{e}). \tag{B.3}$$

The meaning of $\mathfrak{T}$ is that it tells one moves from one Cauchy surface to the other across the foliation.

Now let n be the vector field of normal vectors to the Cauchy surfaces. The field $\mathfrak{T}$ decomposes into two quantities. To see this, perform the orthogonal decomposition with respect to n, i.e., write $\mathfrak{T}$ as a sum of a parallel piece with respect to n and one orthogonal piece:

$$\mathfrak{T} = Nn + \mathfrak{N}. \tag{B.4}$$

We call N the *lapse function* and $\mathfrak{N}$ the *shift vector*.

We have worked purposefully without any reference to coordinates to show explicitly that $t, \mathfrak{T}, N, \mathfrak{N}$ are objects defined by the foliation. Coordinates, however, are useful and we now set out to construct coordinates on M adapted to the foliation.

Define $\hat{t} : \mathbb{R} \times \Sigma \to \mathbb{R}$ the projection onto the first factor $\hat{t}(\lambda, q) = \lambda$. Pick coordinates $\hat{x}^i : \mathbb{R} \times \Sigma \to \mathbb{R}$ on Σ. Then $(\hat{t}, \hat{x}^i)$ is a coordinate chart on $\mathbb{R} \times \Sigma$. It induces a coordinate chart on M by the obvious procedure of defining $t = \hat{t} \circ \varphi^{-1}$ and $x^i = \hat{x}^i \circ \varphi^{-1}$. Notice that the t coordinate is nothing but the time function of the foliation. We can furthermore show that in this coordinate system the time vector field is

$$\mathfrak{T} = \frac{\partial}{\partial t}. \tag{B.5}$$

This follows basically from definition. The vector field ∂_t is defined to act on functions by differentiating along its coordinate lines, but these lines are exactly the curves we used to define $\mathfrak{T}$ in the first place.

We finally study the metric in these coordinates. We shall compute two metric coefficients g_{tt} and g_{ti}. The first is

$$\begin{aligned} g(\partial_t, \partial_t) &= g(\mathfrak{T}, \mathfrak{T}) \\ &= g(Nn + \mathfrak{N}, Nn + \mathfrak{N}) \\ &= N^2 g(n, n) + g(\mathfrak{N}, \mathfrak{N}). \end{aligned} \tag{B.6}$$

but recall that n is the normal to the Cauchy surfaces, which we assume normalized, so that $g(n, n) = -1$ since it is timelike. Furthermore, by definition $\mathfrak{N}$ is orthogonal to it, which justifies dropping the terms we dropped. Finally, $\mathfrak{N}$ is tangent to Σ and hence it has only components along ∂_i. This allows us to conclude

$$g_{tt} = -N^2 + g_{ij} N^i N^j. \tag{B.7}$$

The second term is

$$\begin{aligned} g(\partial_t, \partial_i) &= g(\mathfrak{T}, \partial_i) \\ &= g(Nn + \mathfrak{N}, \partial_i) \\ &= g(\mathfrak{N}, \partial_i) \\ &= g_{ij} N^j, \end{aligned} \tag{B.8}$$

where we have used that ∂_i is orthogonal to n because it is by definition tangent to Σ. All this combined means that in this system of coordinates we can write the metric tensor as

$$\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
&= g_{tt} dt^2 + g_{ti} dt dx^i + g_{ij} dx^i dx^j \\
&= (-N^2 + g_{ij} N^i N^j) dt^2 + g_{ij} N^j dt dx^i + g_{ij} dx^i dx^j \\
&= -N^2 dt^2 + g_{ij} (N^i N^j dt^2 + N^j dt dx^i + dx^i dx^j) \\
&= -N^2 dt^2 + g_{ij} (N^i dt + dx^i)(N^j dt + dx^j). \qquad \text{(B.9)}
\end{aligned}$$

The conclusions we draw from this are: in a globally hyperbolic spacetime we can always foliate spacetime by Cauchy surfaces. A foliation gives rise to a time function, a lapse function and a shift vector, which together comprise the time vector of the foliation.

We can introduce coordinates adapted to the foliation by choosing the time function as one coordinate and completing the chart by adding coordinates on the Cauchy surface. When such a system of coordinates is used the metric tensor decomposes as we have derived. On the other hand, it should be clear that specifying in a globally hyperbolic spacetime a time function and a time vector - or else a lapse function and a shift vector - specifies a foliation by Cauchy surfaces.

C

Radial Null Geodesics of Schwarzschild

In this appendix we discuss the radial null geodesics of Schwarzschild, which shows how light travels. These are the worldlines of incoming and outgoing photons moving at fixed angles. These curves are defined by three conditions:

1. They are null curves $\gamma : I \subset \mathbb{R} \to M$, meaning that they satisfy $g(\gamma', \gamma') = 0$;
2. They are radial curves, meaning that $\theta \circ \gamma = \theta_0$ and $\phi \circ \gamma = \phi_0$ where $(\theta_0, \phi_0) \in S^2$ is a *fixed direction*;
3. They are geodesics, and so satisfy the geodesic equation;

The radial condition implies that such a curve is specified by *two functions*, namely, $t \circ \gamma$ and $r \circ \gamma$. We shall employ the usual abuse of notation and simply denote these functions by t, r leaving the composition implicit.

Now, by inspection of the metric, the Lagrangian for this geodesic in Schwarzschild coordinates is simply

$$L = \frac{1}{2}\left(-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right), \quad f(r) = 1 - \frac{2M}{r}. \tag{C.1}$$

It is clear that since the Lagrangian doesn't depend on t we shall have

$$\frac{\partial L}{\partial t} = 0, \tag{C.2}$$

which implies the t equation is simply

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{t}} = 0. \tag{C.3}$$

This equation gives rise to a constant of motion. One *equivalent* way to see this is to observe that the vector ∂_t is a Killing vector field which renders $g(\partial_t, \gamma')$ a constant of motion when γ is affinely parameterized. Expliciting the inner product, we have:

$$\frac{d}{d\lambda}\left(-f(r)\dot{t}\right) = 0. \tag{C.4}$$

This in turn implies there is a constant $k \in \mathbb{R}$ such that

$$-f(r)\dot{t} = k. \tag{C.5}$$

In a sense, this already reduces the number of unknown functions from two to one, since knowing r completely determines t from the above equation. We could of course proceed to analyze the next geodesic equation, but since we have now just one unknown we can equivalently study the null condition which still has to be imposed.

The condition is $g(\gamma', \gamma') = 0$. In the coordinate system we are working, it is written as

$$-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 = 0. \tag{C.6}$$

This implies, upon using our relation for $\dot{t}$, that the null condition is

$$\dot{r}^2 = f(r)^2\dot{t}^2 = kf(r)^2 f(r)^{-2} = k. \tag{C.7}$$

Eq. (C.7) can be solved immediately to give $r(\lambda) = \alpha\lambda + \beta$ where $\alpha = \pm k$ and β is some other constant. We have to observe that the solutions on this point divide on two categories: the ones for which $k = \alpha = 0$ and hence have r constant, and the ones for which $k \neq 0$ and r varies.

C.1 Ingoing and Outgoing Radial Null Geodesics

We shall study the second case - that with non-constant r - first and shall see that what we get are the so-called ingoing and outgoing radial null geodesics representing wordlines of massless particles radially coming in from far away or radially going out far away. The fact that $r(\lambda) = \alpha\lambda + \beta$ with $\alpha \neq 0$ now implies that it is possible to parameterize the curve by r, so that r is an affine parameter. What we do, is to define a new curve $\tilde{\gamma}$ by the parameterization

$$\tilde{\gamma}(\lambda) = \gamma\left(\frac{1}{\alpha}(\lambda - \beta)\right), \tag{C.8}$$

the reparemeterization is affine, so that $\tilde{\gamma}$ still is an affinely parameterized geodesic. The coordinates along this curve will be denoted by $\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi}$. It follows that

$$\tilde{r}(\lambda) = r\left(\frac{1}{k}(\lambda - c)\right) = k\frac{1}{k}(\lambda - c) + c = \lambda. \tag{C.9}$$

Since $\tilde{r}$ is actually $r \circ \tilde{\gamma}$ this means that the coordinate r is the curve parameter, and is an affine parameter!

We now turn to $\tilde{t}$. Its equation follows easily from Eq. (C.5). Indeed it gives

$$\dot{\tilde{t}} = \frac{d}{d\lambda} t\left(\frac{1}{k}(\lambda - c)\right) = \frac{1}{k}\dot{t}. \tag{C.10}$$

We thus have to solve

$$\dot{\tilde{t}} = \pm f(r)^{-1} = \pm \frac{\lambda}{\lambda - 2M}. \tag{C.11}$$

The above equation can be immediately integrated, in particular using integration by parts, to get

$$\tilde{t} = \pm(\lambda + 2M \ln|\lambda - 2M| - 2M + C), \tag{C.12}$$

where C is a constant, which includes of course specification of initial data. Still, the expression has the problem of taking the logarithm of a dimensionful constant. We can use C to solve this problem and even get rid of the $-2M$ term. The appropriate choice of C is

$$C = 2M - 2M \ln 2M + C', \tag{C.13}$$

where C' contains the remaining freedom for initial data. By using this and relabeling C' by C we finally get the full radial null geodesic, for which we shall return to the notation from the start and denote by γ:

$$t \circ \gamma(\lambda) = \pm\left(\lambda + 2M \ln \frac{|\lambda - 2M|}{2M}\right) + C, \tag{C.14a}$$

$$r \circ \gamma(\lambda) = \lambda, \tag{C.14b}$$

$$\theta \circ \gamma(\lambda) = \theta_0, \tag{C.14c}$$

$$\phi \circ \gamma(\lambda) = \phi_0. \tag{C.14d}$$

We see from Eq. (C.14a) above that the solutions with non-constant r divide once again in two categories, those corresponding to the $+$ sign and those corresponding to the $-$ sign. We can better understand this as follows: recall first that the affine parameter of the geodesics in this construction is r, which is monotonically increasing with the areas of the spheres which are orbits of the rotations. In that sense it is obvious that as r gets bigger one is getting far away.

Now, focus on the region $r > 2M$. Since $(t \circ \gamma)'(\lambda) = \pm \frac{\lambda}{\lambda - 2M}$ and since λ is actually r which in this region is bigger than $2M$ we immediately see that the sign of this derivative is exactly $\pm$. Now, we also know that

$$g(\partial_t, \gamma'(\lambda)) = -f(\lambda)(t \circ \gamma)'(\lambda), \tag{C.15}$$

therefore the sign of this quantity is clearly $\mp$. So for the solutions in the $+$ case we have a minus here, and the curve is *future directed*. So as the parameter increases, one is heading to the future. This means that as time goes on, the massless particle is going far away, and this is one *outgoing null radial curve*. These are plotted in Fig. (C.1):

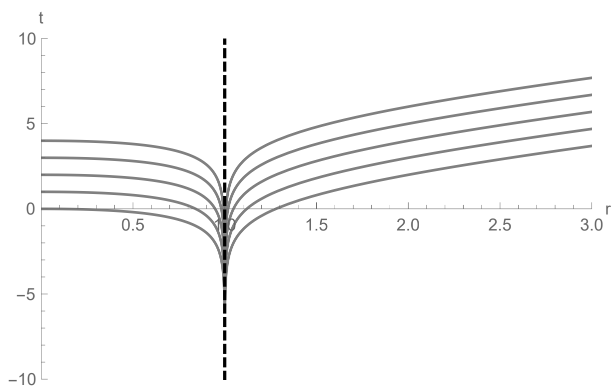

FIGURE C.1: Outgoing radial null geodesics. The dashed vertical lines marks the special value of $r = 2M$ set to $2M = 1$ in this plot.

Similarly, for the $-$ case we have a plus on the $g(\partial_t, \gamma')$ projection, and the curve is *past directed*. The increase of the parameter means one is going to the past, and this can be seen as the worldline of a massless particle which came from far away. Because of that this is one *ingoing null radial curve*. These are also plotted in Fig. (C.2):

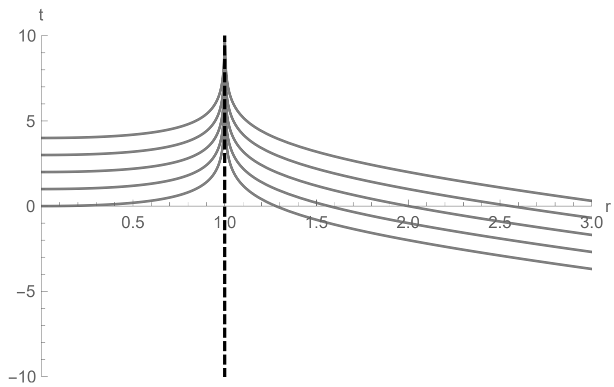

FIGURE C.2: Ingoing radial null geodesics. The dashed vertical lines marks the special value of $r = 2M$ set to $2M = 1$ in this plot.

In both cases there seems to be a very bad behavior at the surface $r = 2M$. For the ingoing solution, the interpretation is that a massless particle falling radially towards $r = 0$, as seen by one Schwarzschild observer, will never actually see the particle pass through $r = 2M$. It will get closer and closer and asymptotically reach this location as $t \to \infty$ for the Schwarzschild observer. Of course this will also happen for massive particles, since such particles are constrained to move slower than massless ones. In any case, it is clear that for the Schwarzschild observer, the location $r = 2M$ is somewhat special in that it represents for it the end of time. For that reason one calls it the event horizon, and denotes it by $\mathcal{H}$.

This bad behavior as one approaches $\mathcal{H}$, on the other hand, can be seen to be just a problem of a bad choice of coordinates. In fact, $\mathcal{H}$ is not a location in the domain of the Schwarzschild chart, so we could not expect a reasonable behavior of a coordinate representation of the lines there. But the fact that there is no true singularity at $\mathcal{H}$ can be seen explicit in our context. This location is approached by either ingoing and outgoing radial null geodesics at *finite affine parameter*, so that this ought to be a real nonsingular location. Furthermore, reaching that location at finite affine prameter means that the particle actually crosses the surface, even if the Schwarzschild observer cannot see this.

The fact that the geometry is well-behaved at the horizon can be seen by choosing coordinates adapted to these radial null geodesics, by which we mean that one of the coordinate lines will coincide with the geodesic. Inspection of Eq. (C.14) which shows the parameterization of the radial null geodesics in Schwarzschild coordinates, suggests the procedure to follow. One first defines the coordinate in the exterior region $r > 2M$,

$$r_* = r + 2M \ln \frac{r - 2M}{2M}, \tag{C.16}$$

which is called the *tortoise coordinate*. Next, one defines a coordinate $u = t - r_*$. It is then immediately clear that the $+$ sign case of Eq. C.14 in these coordinates becomes simply a u constant coordinate line.

In that case, the outgoing null radial geodesics correspond to coordinate lines of r, which is the affine parameter, in the (u, r, θ, ϕ) coordinates. These coordinates are called the *Eddington-Finkelstein outgoing coordinates*. The metric can be transformed accordingly with ease by transforming the basis covectors. Indeed, the only real transformation is

$$du = dt - dr_* = dt - \frac{dr_*}{dr} dr = dt - f(r)^{-1} dr. \tag{C.17}$$

Inverting this to get dt in terms of du and dr, it follows immediately that

$$\begin{aligned} g &= -f(r)(du + f(r)^{-1}dr)^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \\ &= -f(r)(du^2 + 2f(r)^{-1}dudr + f(r)^{-2}dr^2) + f(r)^{-1}dr^2 + r^2 d\Omega^2 \\ &= -f(r)du^2 - 2dudr - f(r)^{-1}dr^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \\ &= -f(r)du^2 - 2dudr + r^2 d\Omega^2, \end{aligned} \tag{C.18}$$

where a key ingredient in the computation is to recall that the product between one-forms appearing in g is the *symmetric tensor product*.

In the same way, defining $v = t + r_*$, it can be seen that the $-$ case of Eq. (C.14) in the new coordinates becomes simply a v constant coordinate line. Thus the ingoing null radial geodesics correspond to coordinate lines of r, which is the affine parameter, in the (v, r, θ, ϕ) coordinates. These coordinates are likewise called the *Eddingtong-Finkelstein ingoing coordinates*. We also write down the metric in these coordinates using the same procedure. We now have

$$dv = dt + dr_* = dt + \frac{dr_*}{dr}dr = dt + f(r)^{-1}dr. \tag{C.19}$$

Again inverting Eq. (C.19) and plugging into the Schwarzschild metric in Schwarzschild coordinates given by Eq. (2.10) we get

$$\begin{aligned} g &= -f(r)(dv - f(r)^{-1}dr)^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \\ &= -f(r)(dv^2 - 2f(r)^{-1}dvdr + f(r)^{-2}dr^2) + f(r)^{-1}dr^2 + r^2 d\Omega^2 \\ &= -f(r)dv^2 + 2dvdr - f(r)^{-1}dr^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 \\ &= -f(r)dv^2 + 2dvdr + r^2 d\Omega^2. \end{aligned} \tag{C.20}$$

So let us summarize the results so far: we observe that what seemed to be a singularity at $\mathcal{H}$ was actually just a defect of the Schwarzschild coordinates. This can be seen by introducing coordinates in which such a surface can be reached at finite affine parameter of geodesics. Using radial null geodesics we found they are affinely parameterized by the Scwarzschild radial coordinate r and constructed two such systems, adapted to these curves. At this point it is already possible to see that the surface $r = 2M$ which we called $\mathcal{H}$ is actually composed of two parts. The part $\mathcal{H}^+$ which is described in the ingoing Eddington-Finkelstein coordinates as the $r = 2M$ surface, and the part $\mathcal{H}^-$ which is described in the outgoing Eddington-Finkelstein coordinates as the $r = 2M$ surface. Up to this point no coordinate chart we built covers both at same time. The Schwarzschild coordinates covers none and the Eddington-Finkelstein coordinates covers just each at a time. With this we conclude the study of radial null geodesics of Schwarzschild with varying r.

C.2 Horizon Generators

We have another set of solutions, though, for which r is constant. We shall study these in analogy to the first case studied in the previous subsection. In the end we shall argue that these curves are in fact the generators of the null surfaces $\mathcal{H}^\pm$.

Let us suppose that we are studying such a solution, and let again t, r denote the coordinates of the curve γ. The null condition, with r constant, becomes

$$-\frac{1}{2}f(r)\dot{t}^2 = 0. \tag{C.21}$$

So, in fact, we only have radial null geodesics with r constant, when $f(r)$ vanishes for this constant value of the radial coordinate, otherwise the curve would not be null. The fact is that $f(r)$ vanishes only for $r = 2M$, which is *not* in the domain of the Schwarzschild chart. In that case, to study this class of solutions, we must go to coordinates on which this location is well represented. We have constructed two such coordinates: the ingoing and outgoing Eddington-Finkelstein coordinates. We set out to study these. Let us work with ingoing Eddington-Finkelstein coordinates first. From Eq. (C.20) we can immediately read off the Lagrangian generating the geodesic equation

$$L = \frac{1}{2}(-f(r)\dot{v}^2 + 2\dot{r}\dot{v} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)). \tag{C.22}$$

The v equation is just one equation for a constant of motion, in fact the same as we had in the previous coordinate system,

$$-f(r)\dot{v} + \dot{r} = k. \tag{C.23}$$

Notice that for the constant r case this is already trivially satisfied with $k = 0$, since $\dot{r} = 0$ and since the null condition demands $f(r) = 0$ for this class of solutions. We therefore study the r equation. It is

$$\ddot{v} + \frac{1}{2}f'(r)\dot{v}^2 - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0. \tag{C.24}$$

Imposing the radial condition, and the $r = 2M$ condition, we immediately get

$$\ddot{v} + \frac{1}{4M}\dot{v}^2 = 0. \tag{C.25}$$

This equation can be solved explicitly and the result is, for $\lambda > 0$, $v(\lambda) = 4M\ln\lambda$. Therefore the radial null geodesics of Schwarzschild with constant value of r in ingoing Eddington-Finkelstein coordinates are:

$$v \circ \gamma(\lambda) = 4M\ln\lambda + C, \tag{C.26a}$$

$$r \circ \gamma(\lambda) = 2M, \tag{C.26b}$$

$$\theta \circ \gamma(\lambda) = \theta_0, \tag{C.26c}$$

$$\phi \circ \gamma(\lambda) = \phi_0. \tag{C.26d}$$

Finally we can perform the study of the outgoing Eddington-Finkelstein coordinates. From Eq. (C.18) we can immediately read off the Lagrangian:

$$L = \frac{1}{2}(-f(r)\dot{u}^2 - 2\dot{r}\dot{u} + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)). \tag{C.27}$$

The u equation again is just the constant of motion equation, which as in the ingoing Eddington-Finkelstein coordinates case, is trivially satisfied on the class of solutions we are interested in. The r equation, however, is in general

$$\ddot{u} - \frac{1}{2}f'(r)\dot{u}^2 + r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0. \tag{C.28}$$

Once the radial condition and the $r = 2M$ condition are imposed we obtain

$$\ddot{u} - \frac{1}{4M}\dot{u}^2 = 0. \tag{C.29}$$

The solution can again be explicitly obtained and in this case it is $u(\lambda) = -4M\ln\lambda$. This means that the radial null geodesics of Schwarzschild with constant value of r in outgoing Eddington-Finkelstein coordinates are

$$u \circ \gamma(\lambda) = -4M\ln\lambda + C, \tag{C.30a}$$

$$r \circ \gamma(\lambda) = 2M, \tag{C.30b}$$

$$\theta \circ \gamma(\lambda) = \theta_0, \tag{C.30c}$$

$$\phi \circ \gamma(\lambda) = \phi_0. \tag{C.30d}$$

We finish by arguing that this class of radial null geodesics with constant radial coordinate, which consequently are neither ingoing nor outgoing, are really generators of the event horizon. Recall that a null hypersurface by definition has a null normal vector. Since the vector is null it is orthogonal to itself, so it is also tangent to the hypersurface. It can then be shown that its integral lines through points of the null surface are geodesics of the ambient spacetime [7]. These geodesics can be affinely reparameterized, and the resulting curves are called the generators of the surface. The affine parameter can then be chosen as one geometrically meaningfull coordinate on the surface.

After this brief explanation, we shall work in Eddington-Finkelstein coordinates (v, r, θ, ϕ). Consider one ingoing null radial geodesic parameterized by r. We can reach $r = 2M$ without problems on it. Let us use these coordinates to discuss the hypersurface so characterized. Its normal one-form is clearly dr. To get a normal vector, we construct the physically equivalent one: $\ell^{\mu} = g^{\mu\nu}(dr)_{\nu}$. In other words, we have $\ell^{\mu} = g^{\mu r}$. This immediately gives

$$\ell = \frac{\partial}{\partial v} + f(r)\frac{\partial}{\partial r}, \tag{C.31}$$

which on the surface $r = 2M$ is

$$\ell\bigg|_{r=2M} = \frac{\partial}{\partial v}. \tag{C.32}$$

This is the normal vector to the surface. It is a null vector on the surface because $\ell_{\mu}\ell^{\mu} = -f(r)$ which is zero at $r = 2M$. This means that the surface is null, and as anticipated, this vector is the tangent to the null generators of the null surface. We seek one affine parameterization of these curves. To do so, we shall solve the geodesic equation, imposing two constraints: the first is that the curve is tangent to ℓ^{μ} and the second is that it is constrained to live inside $r = 2M$.

The first constraint means that the curve is null and radial. The second constraint, means that it has constant r with value $r = 2M$. But this is precisely one of the class of radial null geodesics of Schwarzschild we studied! We already know these solutions, they have $v(\lambda) = 4M \ln \lambda$.

We can also perform the same development in the outgoing Eddington-Finkelstein coordinates (u, r, θ, ϕ). Considering the null surface defined by $r = 2M$ in this system of coordinates we solve the equation for its generators. By exactly the same procedure, we find that they are given by $u(\lambda) = -4M \ln \lambda$.

With this we complete the study and classification of the radial null geodesics of Schwarzschild. They are seen to fall in three categories:

1. The ones with constant r, which are given by Eq. (C.26) and (C.30). These are also seen to be respectively the generators of the null surfaces $\mathcal{H}^+$ and $\mathcal{H}^-$;

2. The ones with varying r which are ingoing given by (v, θ, ϕ) constant in ingoing Eddington-Finkelstein coordinates and affinely parameterized by r;

3. The ones with varying r which are outgoing given by (u, θ, ϕ) constant in outgoing Eddington-Finkelstein coordinates and affinely parameterized by r;

D

The postulates of Quantum Mechanics

Here we shall review the standard postulates of Quantum Mechanics. We follow [9] closely, however we work with the spectral theorem and projective measures [3]:

- The states of a system are unit rays in a separable Hilbert space $\mathscr{H}$, so that they can be represented by unit vectors $|\psi\rangle \in \mathscr{H}$ being equivalent to $e^{i\alpha}|\psi\rangle$ for any phase.
- The quantities one can measure associated to a system are described by hermitian operators on the system's Hilbert space. These operators are called observables.
- The possible values a physical quantity may attain are the elements of the spectrum of the observable.
- Let A be an observable with projective measure $\mathbb{P}_A$ defined on the Borel sigma algebra of its spectrum $\sigma(A)$ by means of the spectral theorem. If the state of the

system is $|\psi\rangle$ the probability that a measurement of A lies in $S \subset \sigma(A)$ is

$$P(S) = \langle\psi|\mathbb{P}_A(S)|\psi\rangle. \tag{D.1}$$

- In the conditions of the previous postulate, if A is measured in state $|\psi\rangle$ and the result lies in $S \subset \sigma(A)$ then the post-selected state after the measurement is the normalized projection
$$|\psi'\rangle = \frac{\mathbb{P}_A(S)|\psi\rangle}{\sqrt{\langle\psi|\mathbb{P}_A(S)|\psi\rangle}}. \tag{D.2}$$

- The time evolution with initial instant t_0 of an isolated system is performed by means of a one-parameter strongly continuous family of unitary operators $U(t, t_0)$ satisfying the Markov property
$$U(t, t')U(t', t_0) = U(t, t_0). \tag{D.3}$$
Thus if $|\psi\rangle$ is the initial state specified at t_0 the state at time t is
$$|\psi(t)\rangle = U(t, t_0)|\psi\rangle. \tag{D.4}$$
Furthermore, $U(t, t_0)$ is generated by the Hamiltonian observable $H(t)$, by means of the dynamical equation
$$i\hbar\frac{dU}{dt}(t, t_0) = H(t)U(t, t_0). \tag{D.5}$$

These are the standard postulates in a mathematically rigorous form. When the spectrum of the operator A is discrete, it is of the form $\sigma(A) = \{a_n : n \in I \subset \mathbb{N}\}$ where I can be either finite or infinite and it has genuine eigenvectors. In other words, for each $\lambda \in \sigma(A)$ there's $|\psi_\lambda\rangle \in \mathscr{H}$ with $A|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle$.

The subspace of $\mathscr{H}$ composed of all $|\psi\rangle$ with $A|\psi\rangle = \lambda|\psi\rangle$ for $\lambda \in \sigma(A)$ is the eigenspace associated to λ, which is usually denoted $\mathscr{H}_\lambda$. The dimension of $\mathscr{H}_\lambda$ is the degeneracy of the eigenvalue. Now for each $n \in I$ there is one eigenvalue a_n and one eigenspace $\mathscr{H}_n$. In that case, $\mathscr{H}$ decomposes as one orthonogonal direct sum

$$\mathscr{H} = \bigoplus_{n\in I} \mathscr{H}_n. \tag{D.6}$$

In that case, the singleton sets $\{a_n\}$ are measurable with non-zero measure and the projectors $\mathbb{P}_A(\{a_n\})$ are just the projector operators onto $\mathscr{H}_n$. In particular, if for one $m \in I$, a_m is non-degenerate so that $\mathscr{H}_m$ has dimension one, the projector is $\mathbb{P}_A(\{a_m\}) = |\psi_m\rangle\langle\psi_m|$, where $|\psi_m\rangle \in \mathscr{H}_m$ is one unit vector spanning the subspace.

Although the continuous case can be handled by von-Neumann's spectral theory, in practice it is common to use Dirac's formalism, which in a sense mimics the discrete case above outlined. The issue with the continuous case is that in said situation there are no eigenvectors. In other words, if $\sigma(A)$ is continuous and one picks $\{x\} \subset \sigma(A)$, this set has measure zero and one cannot define the associated eigenspace and the corresponding eigenvectors. The point of Dirac's formalism is to pretend such eigenvectors do exist. The idea of Dirac's formalism is that for every $x \in \sigma(A)$ one supposes there exists one eigenvector $|x\rangle$ generating a one-dimensional subspace $\mathscr{H}_x$. The projector onto the subspace is denoted $|x\rangle\langle x|$. In that case, the projector-valued integration measure is then denoted conveniently as $d\mathbb{P}_A(x) = |x\rangle\langle x|dx$ where $x \in \sigma(A)$.